LabMath Indonesia

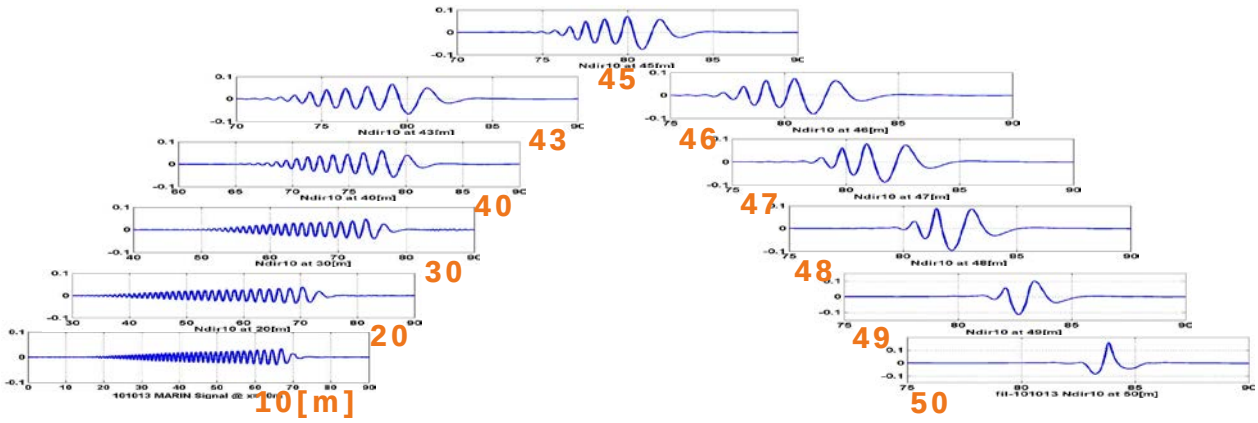
Water Wave Modelling & Simulation

YAB LabMath

With

Introduction to HAWASSI software

www.hawassi.labmath-indonesia.org



Published by:
LabMath-Indonesia
Jl. Dago Giri 99,
Bandung 40391

Version 2016.05

www.labmath-indonesia.org

Contents

Preface

Waves are fascinating, important and challenging.
The importance can be substantiated from some well-known observations:

- Half of the world population lives less than 150 km from the coast
- The sea is a relatively easy medium for transport of people and goods (half of all the world crude oil and increasingly more natural gas) and for intercontinental telecommunication through cables
- Ocean resources of food and minerals are only at the start of discovery, profits from wind parks and harvesting of wave energy in coastal areas is expanding.

Therefore, a sustainable and safe development of the oceanic and coastal areas is of paramount importance. Nowadays that means that for the design of harbours, breakwaters and ships, calculations are performed with increasingly more accurate and fast simulation tools. Tools that are, packaged in software, based on the basic physical laws that describe the properties of waves, the wave-ship interaction, the forces on structures, etc.

HAWASSI software is aimed to contribute to extend the accuracy, capability and speed of existing numerical methods and software using advanced applied-mathematical modelling methods that are at the basis.

A basis with a rich history that is fascinating and challenging. Starting in the 18th century with Euler who generalized Newton's law for fluids, in the 19th century Airy 'solved' the problem to describe small amplitude surface water waves. In that same century, many renowned scientists like Scott Russel, Stokes, Boussinesq, Rayleigh and Korteweg & De Vries investigated the nonlinear aspects of finite amplitude waves. As much as possible without the need to fully calculate the internal fluid motion, started with Boussinesq in an approximative way, this was formulated accurately in the 1960-1970's by Zakharov and Broer by providing the Hamiltonian form of the dynamic equations.

HAWASSI software is based on these last findings, with methods for making the principal description into a practical (numerical) modelling and implementation tool.

Let nature tell its secrets
Listen to the physics in its mathematical language
Restrain from idealization
Only then models will serve us in abundance

This service booklet provides a short introduction to the basics phenomena and description of water waves and the basic underlying methods for the models used in HAWASSI software.

In Chapter 1 a rather extensive description is given of basic concepts such as dispersion, diffraction, shoaling, nonlinearity, spectra, etc. of long crested and short crested

wave phenomena, with the purpose to introduce knowledge of phenomena that is essential in the process of interpreting the results of simulations.

Chapter 2 provides the basic description of irrotational free surface water waves, the classical description in terms of partial differential equations;

Chapters 3 describes the Hamiltonian-Boussinesq approach to surface waves: the formulation as a dynamic system in Hamiltonian form, and the way how Dirichlet's principle is used to arrive to the dimension reduction by modelling, instead of calculating, the interior flow. It also contains the layout of the numerical spatial discretizations that retain the basic Hamiltonian form and guarantees energy conservation, avoiding instabilities.

Chapter 4 specializes to the basic description of the two models that are part of HAWASSI software, the (Optimized) Variational Boussinesq Model and the AB- model.

Appendix A provides a scholarly description of the 'Art' of scientific simulation, and Appendix B a condensed account of variational methods.

The manuals of HAWASSI software contain descriptions of test cases for hands-on simulation with the software that can illustrate the phenomena, and at the same time makes the user familiar to deal with the software in a critical and responsible way.

References to literature and original sources are given.

We add with pleasure that this is work in progress: our understanding and learning together with others continues and will hopefully get appropriate description in new versions of this booklet and improved and in extended versions of HAWASSI.

May 2016,
Andonowati & Brenny van Groesen

ACKNOWLEDGEMENT 1. Thanks to the institute LabMath -Indonesia, Bandung Indonesia, and the group Applied Analysis & Mathematical Physics at the University of Twente, Netherlands, for the opportunities and technical support to execute this research at these two places in the past 10 years in which the HAWASSI software was developed.
A very strong motivation for the work arose from collaboration with the hydrodynamic Laboratories MARIN (Maritime Research Institute Netherlands, Wageningen) and in earlier stage with LHI (Indonesian Hydrodynamic Laboratory, Surabaya). We thank Prof.dr. Huijsmans (now at TUDelft), Dr. Tim Bunnik (MARIN) and the late Dr. Andi Jamluddin (LHI) of these institutes for the pleasant and fruitful discussions over many years and the use of their data to test and improve our software.
We thank many students and scientists who worked during these past 10 years (some up till this day) on various parts of the wave models, often with the aim to obtain a PhD-degree; in a more or less chronicle order: Gert Klopman, Natanael Karyanto, Ivan Lakhturov, Lie She Liam, Didit Adytia, Wenny Kristina, Arnida Latifah, Ruddy Kurnia, Mourice Woran, Andreas Parama Wijaya, M.Hafiizh, Abrari Noor, Peri Turnip.

ACKNOWLEDGEMENT 2. Funding for the research activities was provided by the institutes and by various grants from diverse organizations over the past ten years:
STW, The Technical Division of NWO;
KNAW, the Royal Netherlands Academy of Arts and Sciences,
NWO, The Netherlands Organization of Scientific Research;
WOTRO, Division of NWO Earth and Life Sciences;
APN, Asian Pacific Network, Capable Project

ACKNOWLEDGEMENT 3. The geometric stripes in the HAWASSI-logo were designed by a *Homo erectus*, more than 400.000 years ago. A freshwater shell with such engravings was found by Eugene Dubois in 1891 near the city of Trinil, Central Java, Indonesia, (the place where also the Java man of some 1.5 year old was discovered). Preceding later engravings with some 300.000 years, we can only guess that possibly the patterns are an expression of fascination for the waves that were encountered and observed with attention during their collection of shells.

Ref: J.C.A. Joordens ea.,
Homo erectus at Trinil on Java used shells for tool production and engraving,
Nature 518, 228–231, 2015 (doi:10.1038/nature13962)

CHAPTER 1

Basic wave phenomena and descriptions

In this chapter we introduce the basic concepts of water waves. A sufficient understanding of these physical concepts, and the translation to mathematical formulations, will be essential for the understanding of the consecutive chapters. While a selection can be made in reading the next chapters, it is advised to read the full contents of this first chapter to obtain the mental infrastructure that will facilitate the reading and reasoning in the next chapters. Readers with sufficient background, will be able to digest the contents in a minimal amount of time. The way of writing in this chapter is aimed to introduce less mathematically mature readers into the formal formulations of gradually increasing mathematical-physical methods later-on.

1. Heuristics

1.1. Introduction. Waves at sea are fascinating to watch from a beach or ship. For calm seas, the seemingly monotone wave patterns still show large variations in wave height and in time and distance between waves. Long crested waves are accompanied and disrupted by much shorter waves which can arise and disappear any moment. Near beaches, wave heights can grow substantially and may break to form waves that are spectacularly used by surfers. During storms, waves can become very high and develop foamy crests; the abundant dynamics doesn't leave the patterns unaltered for even a second. High waves during storms or caused by the bathymetry or by collision against constructions (off-shore or onshore against break waters) can do great harm to ships, the constructions and to people living near the coast.

The fascination and the practical relevance have been the stimuli for extensive studies on water waves, probably as long as people live on earth but certainly in the past centuries. Even more so, the scientific development in physics and mathematics since the 17th century has been stimulated considerably by the investigations of water waves. Despite large successes and great achievements, until this moment still many problems remain to be solved; problems at a basic level of understanding and modelling, as well as problems that arise from ever-increasing new technical demands, such as predictions of waves at a ship from radar observations in real time that can support sea operations for the off-shore industry with oil-gas activities, helicopter landings, etc. Improved design of harbours and breakwaters for beach protection can profit from further improvements as well.

The key to scientific development are improved qualitative mathematical-physical modelling of the complicated processes underlying the water wave dynamics. To investigate the models, and to obtain quantitative information from them, most times numerical implementations of the models are required. *Efficient and reliable numerics should go hand in hand and become part of the modern modelling process.*

These lecture notes aim to serve an initial understanding of the main notions needed to discuss about waves in the ocean and seas and to explain the background of some of present models.

1.2. Geometric-Kinematic description. Looking at the waves approaching a beach, we can illustrate the most important aspects of water waves. We observe the water elevation, relative to some zero level in the absence of waves. We will denote the elevation by $\eta(x, y, t)$ where x, y are the two horizontal coordinates (say x towards the coast, y along the shore), and t the time.

Taking a picture, a snapshot, we see the water surface, consisting of a large combination of distinguishable waves: the wave crests and troughs more or less parallel to the coast line. Looking in the x-direction towards the ocean, we see a succession of wave crests and troughs, from more or less regular patterns far away, to breaking waves closer to the coast, to smaller waves up to the shore line. This snapshot, say at $t = t_0$, is described by the spatial function $(x, y) \to \eta(x, y, t_0)$, the *wave profile*.

On the other hand, we can imagine to stand in the water at a fixed position (x_0, y_0), and experience the changing water height as function of time: $t \to \eta(x_0, y_0, t)$, which is the *wave signal* at that position. The dynamics causes a continuously changing pattern of the complete wave field: $t \to \eta(x, y, t)$.

Characteristic of the above description is the recognition of the difference between the wave field pattern (profile) and the dynamics. Let us write this in a very simplified

way, restricting to one spatial variable, x, in the off-shore direction. If we denote the wave profile at a certain moment by $x \to p(x)$, the simplest possible dynamics would be a translation of that profile with a constant velocity c. That translation is described by

$$\eta(x,t) = p(x - ct)$$

and shows that the spatial profile can be recovered by the time signal at a fixed position. This type of translational wave motion describes approximately the incoming waves far out of the coast; then approximately the wave profile has some periodic pattern. The simplest periodic profile is described by a harmonic function like $\sin(kx)$ where k is the *wavenumber*, related to the *wavelength* λ according to $k = 2\pi/\lambda$. Then the translation with speed c is given by

$$\eta(x,t) = \sin(k(x - ct)).$$

The time signal at a fixed point is sinusoidal also, and periodic in time with *period* T given by $T = 2\pi/\omega$, where ω is the *frequency*, in this example given by $\omega = ck$. This shows that for such so-called *harmonic waves*, there is a relation between spatial wavelength and time period determined by the velocity; this relation is called the *dispersion relation* in the following. The velocity c is called the *phase speed*, since when shifting with speed c (a straight line in the x,t-plane), we follow the wave crest or wave trough.

1.3. Physical ingredients. The above is a purely mathematical description of a simple wave phenomenon, but cannot determine what the actual speed is. To find this out we need a more precise description of the physics underlying the wave motions. This will be done extensively in the following. Here we will just give a simple dimensional argument to illustrate the basic physics.

The physical dimension of a speed is length per time unit, say meter per second,

$$[c] = m/s$$

and we look for others parameters in the wave phenomenon that can lead to this dimension. There are two obvious length scales, the depth of the water, denote this by D, and

the wavelength λ; both have dimension of length:

$$[\lambda] = m, [D] = m$$

Clearly we need another parameter that contains the time. This is given by the driving force of wave propagation, which is the *gravity*. Indeed, in the absence of gravitation, there would be no obstruction to have a hump of water that remains stationary; gravitation will try to equalize the water surface, providing the driving force for changes in the water surface.

Denoting by g the gravitational constant, g has the dimension

$$[g] = m/s^2$$

These seem to be the only parameters that have to be taken into account, when ignoring viscosity effects. Hence we get the possible relations

$$c \sim \sqrt{gD} \text{ or } c \sim \sqrt{g\lambda}$$

This simple dimensional analysis is quite successful in this case, since both results are valid in two (opposite) limiting cases. Indeed, in the next Section we will see that it turns out that for waves of small amplitude (in the linear limit), it holds that

$$c = \sqrt{gD} \text{ for long waves } (\lambda/D >> 1), \text{ the } \textit{shallow water limit} \tag{1.1}$$

$$c = \sqrt{g\lambda/(2\pi)} \text{ for waves on deep water } (\lambda/D << 1), \text{ the } \textit{deep water limit} \tag{1.2}$$

As one result, we can conclude that the example of the translation of a wave profile with constant speed can only be expected to hold above a flat bottom. When the bottom is not flat, the bathymetry $D = D(x)$ will determine the change of the propagation speed, for long waves like

$$c(x) = \sqrt{gD(x)} \tag{1.3}$$

The changing speed will lead to changes in wavelength and in amplitude, as we will see.

1.4. Math-physical description with pde's. We will show and derive the governing wave equations in Chapters 2 and 3 based on physical considerations, like mass conservation and Newtons law of motion, and the more sophisticated formulation with variational principles. Just like dynamical systems from Classical Mechanics, with the pendulum equation and the harmonic oscillator as prime examples, we are used to describe the dynamics with *differential equations*. The same will be true for the wave dynamics: first and second order derivatives with respect to time of the surface elevation will appear. But, different from the discrete mass systems of classical mechanics, we are now dealing with a fluid, a *continuous system*, in which the particle index is replaced by a continuous space variable, like x above. Interactions between neighbouring 'particles' are then expressed using the derivative with respect to x. The relation between derivatives with respect to time and space of one quantity are called *partial differential equations* (pde's). We will show the simplest example of a pde here, which is related to the simplest wave phenomenon, i.e. the *translation equation*. To that end it is noted that it can be observed that the uniform translation with velocity c, described by $\eta(x,t) = p(x-ct)$, satisfies the so-called translation equation, given by

$$\partial_t \eta = -c\partial_x \eta. \tag{1.4}$$

Conversely, any solution of this equation is given by $\eta(x,t) = f(x-ct)$ for some function f. More precisely, if the initial profile, at $t = 0$, is given,

$$\eta(x,0) = f(x),$$

we find uniquely the solution $\eta(x,t) = f(x - ct)$.

The above pde is the simplest wave equation, and illustrates that when having derived wave equations from physical principles as pde's there is the possibility to look for solutions that satisfy specific initial conditions.

The more general second order pde

$$\partial_t^2 \eta = c^2 \partial_x^2 \eta, \tag{1.5a}$$

which can be written like $(\partial_t - c\partial_x)(\partial_t + c\partial_x)\eta = 0$, allows both waves running to the right and to the left, and has as general solution

$$\eta(x,t) = f(x - ct) + g(x + ct)$$

for arbitrary functions f and g which may be determined by prescribing η and the velocity of the surface $\partial_t \eta$ as functions of x (i.e. at each place) at an initial time.

2. 1HD Harmonic waves and dispersion

The simplest possible example of a 'wave' (illustrating a profile at fixed time, a time signal at fixed position, and a translation in space with increasing time) is a harmonic wave

$$A\cos\left(kx-\omega t+\theta\right) \tag{1.6}$$

with A the (real) amplitude and θ the (real) phase of the wave.

To formulate the following more concisely, we introduce complex notation and consider *harmonic modes* with wave number k and frequency ω :

$$a\exp i\left(kx-\omega t\right) \tag{1.7}$$

The wavenumber and frequency are related to the wave length λ and the period T according to

$$k=\frac{2\pi}{\lambda},\omega=\frac{2\pi}{T} \tag{1.8}$$

The amplitude a will be complex in general, and includes information about the phase θ; in the example above

$$a=A\exp\left(i\theta\right)$$

By writing $\exp i\left(kx-\omega t\right)=\exp ik\left(x-ct\right)$ with $c=\omega/k$, we observe that the mode translates with the phase speed ω/k.

Depending on the problem at hand, there is a specific relation between k and ω to guarantee that such modes are solutions; that relation is called the *dispersion relation.* For the simple case of the translation equation, $\partial_t\eta+c_0\partial_x\eta=0$, we saw already that the dispersion relation is $\omega=c_0k$. This relation between k and ω is linear; when that is the case we call the equation non-dispersive.

In most cases, this dispersion relation $\omega=\omega\left(k\right)$ will depend nonlinearly on k, i.e. the phase velocity will not be constant but depend on the wavenumber:

$$c=c\left(k\right)=\frac{\omega\left(k\right)}{k} \tag{1.9}$$

then we talk about *dispersive wave equations.*

In particular, for water waves to which we will restrict ourselves, the *wave dispersion* for infinitesimal waves is given by the following remarkable formula:

$$\boxed{\omega=\Omega\left(k,h\right)=c_0k\sqrt{\tfrac{\tanh(kD)}{kD}},\text{with } c_0{=}\sqrt{gD}} \tag{1.10}$$

Hence, the harmonic modes are given by

$$a_{\pm}\exp i\left(kx\pm\Omega\left(k\right)t\right)$$

with the $+$ sign is for waves running to the left, and the $-$ sign for waves running to the right.

The explicit formula for Ω can be well understood, as we will see in Chapter 3. Observe the limiting cases

$$\Omega\left(k,D\right)\rightarrow\begin{cases}c_0k & \text{for}\quad kD\rightarrow0\\ k\sqrt{g/\left|k\right|} & \text{for}\quad kD\rightarrow\infty\end{cases} \tag{1.11}$$

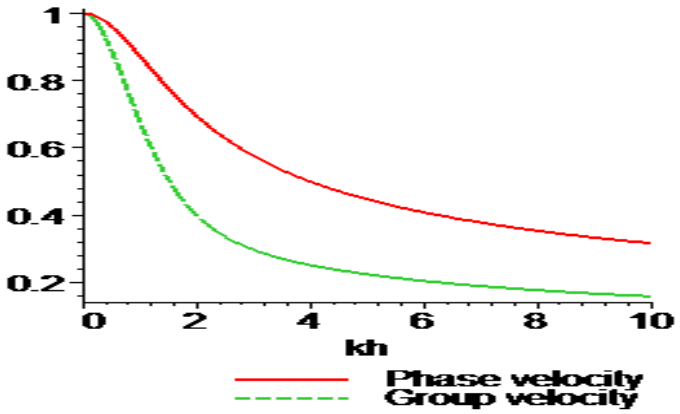


FIGURE 1. Plots of the phase speed (upper curve) and the group speed (lower curve), normalized by $c_0 = \sqrt{gh}$, as function of kh.

The graph of the function $k \to \Omega(k)$ for finite depth is a concave curve, with slope c_0 near $k = 0$ and $\sim \sqrt{gk}$ for $k \to \infty$. The *phase velocity* is given by

$$C(k, D) = \frac{\Omega}{k} = c_0 \sqrt{\frac{\tanh(kD)}{kD}} \tag{1.12}$$

leading to the expressions for the phase velocity that were mentioned in the previous section:

$$C(k, D) \to \begin{cases} \sqrt{gD} & \text{for} \quad kD \to 0 \text{ (shallow water)} \\ \sqrt{g/|k|} = \sqrt{g\lambda/(2\pi)} & \text{for} \quad kD \to \infty \text{ (deep water)} \end{cases} \tag{1.13}$$

The *group velocity* is defined as the derivative of Ω, and has limiting values as

$$V(k, D) = \partial_k \Omega(k, D) \to \begin{cases} \sqrt{gD} & \text{for} \quad kD \to 0 \text{ (shallow water)} \\ \frac{1}{2} C(k) & \text{for} \quad kD \to \infty \text{ (deep water)} \end{cases} \tag{1.14}$$

Plots of the normalized velocities $C(k, D)/c_0, V(k, D)/c_0$ as function of kD are given in Fig. 1.

Observe that the phase and group velocity decrease with increasing kD, so for decreasing wavelength. The speed c_0 is the maximal speed for long waves; for shorter waves the speed monotonically decays to zero.

3. Special phenomena

3.1. Waves over bathymetry. To get a feeling for the dependence on depth, Fig. 2 shows curves for different values of D for wavelength, group and phase velocity as function of the period, and Fig 3 the wave length for period $T = 12$ as function of depth. In Fig 4 for a single linear mode from the deep ocean to the shallow coastal area, during which the frequency is fixed, the phase velocity decays monotonically during the transition, but the group velocity shows a non-monotone behaviour.

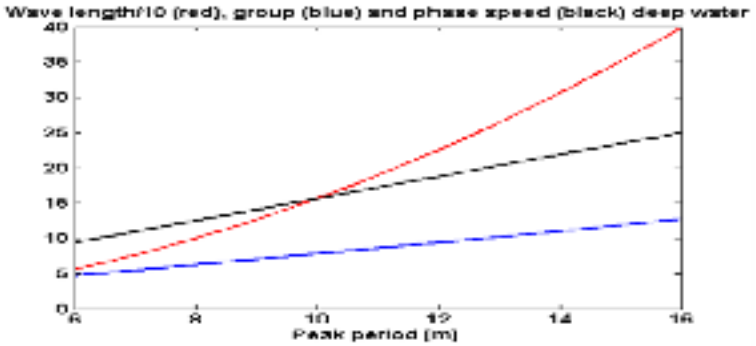


FIGURE 2. Wave length (red), group (blue) and phase velocity (black) as function of peak period.

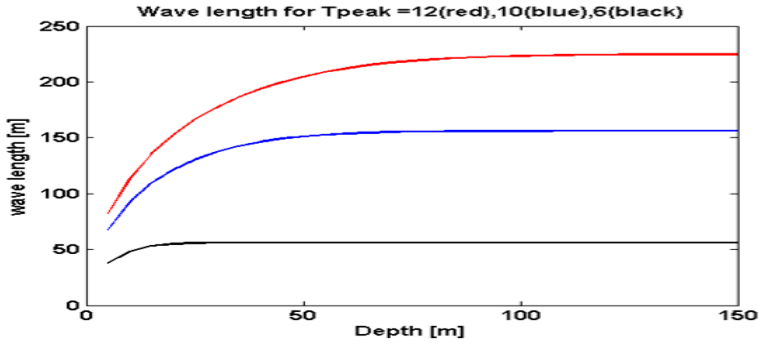


FIGURE 3. The wave length as functio of depth for period T=12 (red), 10 (blue) and 6 (black)

Shoaling

For a harmonic wave running from deep to shallow water the frequency remains fixed. Hence the wave length changes such that

$$\omega = \Omega\left(k\left(x\right), D\left(x\right)\right) \text{ constant} \tag{1.15}$$

or

$$k\left(x\right) = \Omega^{-1}\left(\omega, D\left(x\right)\right) \tag{1.16}$$

In the limit for long waves this is simply $\omega = c\left(x\right) k\left(x\right)$ with $c\left(x\right) = \sqrt{gD\left(x\right)}$, hence $k\left(x\right) = \omega / c\left(x\right)$, and the wave length decreases with the square root of the depth when running in shallower water:

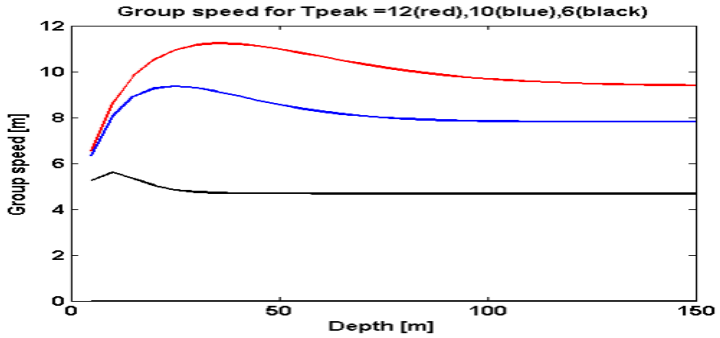


FIGURE 4. Group velocity as function of depth for different period T=12 (red), 10 (blue) and 6 (black)

$$\frac{\lambda(x)}{\lambda_0} = \frac{c(x)}{c_0} = \sqrt{\frac{D(x)}{D_0}} \text{ for long waves} \tag{1.17}$$

But this has also a consequence for the amplitude of the wave. In order to measure this, we introduce the energy of the wave profile over one wave length L: for $\eta = a\sin(kx)$ this energy E is related to

$$E = \int_0^L a^2 \sin^2(kx)\, dx = \frac{a^2 L}{2} \tag{1.18}$$

Conservation of this variance implies that the amplitude will depend on the depth, $a = a(x)$, according to

$$\lambda(x).a^2(x) = \text{ constant.} \tag{1.19}$$

Hence the amplitude increases according to

$$\frac{a^2(x)}{a^2(D_0)} = \frac{\lambda_0}{\lambda(x)} = \frac{\Omega^{-1}(\omega, D(x))}{\Omega^{-1}(\omega, D_0)}$$

For long waves the increase is found to be inversely proportional to the fourth root of the depth:

$$\frac{a(x)}{a(D_0)} = \left[\frac{D_0}{D(x)}\right]^{1/4}. \tag{1.20}$$

Observe that according to this formula $a(x) \to \infty$ for $D(x) \to 0$. Nonlinear effects (including breaking) and bottom friction (which decreases the energy) will dominate the behaviour for shallow depths, so that this limit is not realistically obtained. The effect of amplitude increase when running in shallower water is called shoaling.

3.2. Bi-harmonic waves. Superposition of many waves leads quickly to complicated wave patterns, as will be shown below. Here we consider a special case to illustrate the concept of group speed.

Consider the superposition of two harmonics of the same amplitude with (slightly) different wave lengths

$$\eta_{BH} = \sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t) \tag{1.21}$$

Using a simple identity from geometry we can rewrite this as

$$\eta_{BH} = 2\sin\left(\bar{k}x - \bar{\omega}t\right)\cos\left(\left(\Delta k\right)x - \left(\Delta\omega\right)t\right) \tag{1.22}$$

with

$$\bar{k} = \frac{1}{2}\left(k_1 + k_2\right), \bar{\omega} = \frac{1}{2}\left(\omega_1 + \omega_2\right)$$
$$\Delta k = \frac{1}{2}\left(k_1 - k_2\right), \Delta\omega = \frac{1}{2}\left(\omega_1 - \omega_2\right)$$

The sum is a multiplication of two waves

$$\begin{aligned}\text{carrier wave } &= \sin\left(\bar{k}x - \bar{\omega}t\right)\\ \text{envelope wave } &= 2\cos\left(\left(\Delta k\right)x - \left(\Delta\omega\right)t\right) = 2\cos\left(\left(\Delta k\right)\left[x - \frac{\Delta\omega}{\Delta k}t\right]\right)\end{aligned}$$

Especially for small Δk, and hence small $\Delta\omega$, it shows that the sum is a *modulation* of the carrier wave; the modulation is an envelope wave that has long wave length ($\sim 1/\Delta k$) with the form of a beat pattern. Different from the carrier wave, the envelope has a speed that is (approximately) equal to the group speed: $\Delta\omega/\Delta\kappa \approx \partial_k\Omega\left(k, D\right)$. (See Exercises)

3.3. Boundary conditions. Until now we have reasoned as if there is no limitation on the spatial domain. In reality, of course, we will have to deal with waves in bounded domains. This implies that at the boundary we need to specify the characteristics of the boundary by so-called *boundary conditions*. A physical boundary is for instance the sea-land boundary at the shore line. But when designing a numerical domain, often we will have artificial numerical boundaries in the ocean to reduce the computational domain. These boundaries should be such that incoming waves (from the exterior part of the ocean) can be fluxed into the numerical domain, while, on the other hand, reflected waves approaching the boundary from the interior should not be affected by this artificial boundary; *transparent-influx boundary conditions* are then needed. In fact, the problem to define suitable boundary conditions, and their consistent numerical implementation, is still a problem that is often more difficult than the derivation and implementation of the internal equations.

3.3.1. *Full and partial wave reflection.* As an example of boundary conditions, consider a wave above a flat bottom approaching from the left a *hard wall* at $x = 0$. Taking as model equation the second order wave equation,

$$\partial_t^2\eta = c^2\partial_x^2\eta, \text{ for } x > 0 \tag{1.23}$$

suppose that the incoming wave is given by $F\left(x - ct\right)$, for some function F. At the hard wall, we require the wave to be (perfectly) reflected, which corresponds to the boundary condition

$$\partial_x\eta\left(0, t\right) = 0 \tag{1.24}$$

Then the total solution is given by

$$\eta\left(x, t\right) = F\left(x - ct\right) + F\left(-x - ct\right) \text{ for } x > 0 \tag{1.25}$$

which includes the full reflection at $x = 0$.

In particular, for an incoming harmonic wave from the left

$$\eta_{in} = a\sin\left(kx - \omega t + \phi\right)$$

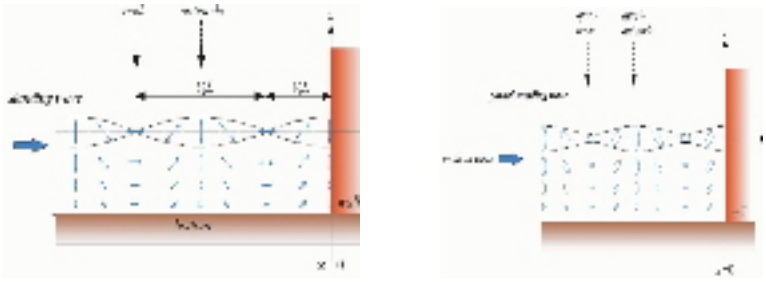
bouncing against a wall at position $x = 0$, the fully reflected wave is given by

$$\eta_{refl} = a \sin(-kx - \omega t + \phi)$$

so that the resulting elevation is given before the wall $x < 0$ by

$$\begin{aligned}\eta_{in} + \eta_{refl} &= a\left[\sin(kx - \omega t + \phi) + a \sin(-kx - \omega t + \phi)\right] \\ &= 2a \cos(kx) \sin(-\omega t + \phi)\end{aligned}$$

which is a *standing wave*: the profile $2a\cos(kx)$ is stationary but its amplitude changes in harmonically in time according to $\sin(-\omega t + \phi)$.



For a partially reflecting harmonic wave with reflection coefficient R the elevation can be written as

$$\begin{aligned}\eta_R &= (1 - R)\, a \sin(kx - \omega t + \phi) + 2Ra \cos(kx) \sin(-\omega t + \phi) \text{ for } x < 0 \\ &= (1 - R)\, a \sin(kx - \omega t + \phi) \text{ for } x > 0\end{aligned}$$

3.3.2. *Transparent boundary conditions.* On the other hand, suppose we want the point $x = 0$ to be an artificial boundary that has to be completely transparent; this is for instance desired if we only want to calculate in the interval $x < 0$, while the physical domain is actually the whole real line. Then a wave from the left travelling to the right, like $F(x - ct)$, should not be disturbed by the boundary at $x = 0$. This can be achieved by requiring the *transparent boundary condition* given by

$$(\partial_t + c\partial_x)\, \eta = 0 \text{ at } x = 0 \tag{1.26}$$

If we take

$$(\partial_t + c\partial_x)\, \eta = \psi(t) \text{ at } x = 0 \tag{1.27}$$

the boundary $x = 0$ is still transparent for to the right travelling waves, but at the same time describes an influx that generates a wave travelling to the left given by

$$\eta(x, t) = \Psi(t + x/c)$$

where Ψ is related to ψ according to $2\Psi'(t) = \psi(t)$.

In more general cases, such as dispersive waves consisting of a superposition of harmonic waves with different wave length, or nonlinear waves, or waves above varying bottom, derivation and implementation of such boundary conditions can be much more difficult, making it an area of ongoing research.

3.4. Dispersive wave (de-)focussing. The effect of dispersion has many consequences. Take as *initial value* for the wave form a positive 'hump', such as for instance a Gaussian profile. Then since the Fourier transform of a Gaussian is a Gaussian also, the initial profile consists of waves of any length. The evolution of the hump for $t > 0$ shows that the Gaussian profile falls apart in two deforming profiles running in opposite directions. Consider the waves travelling to the right. The wave modes with longest

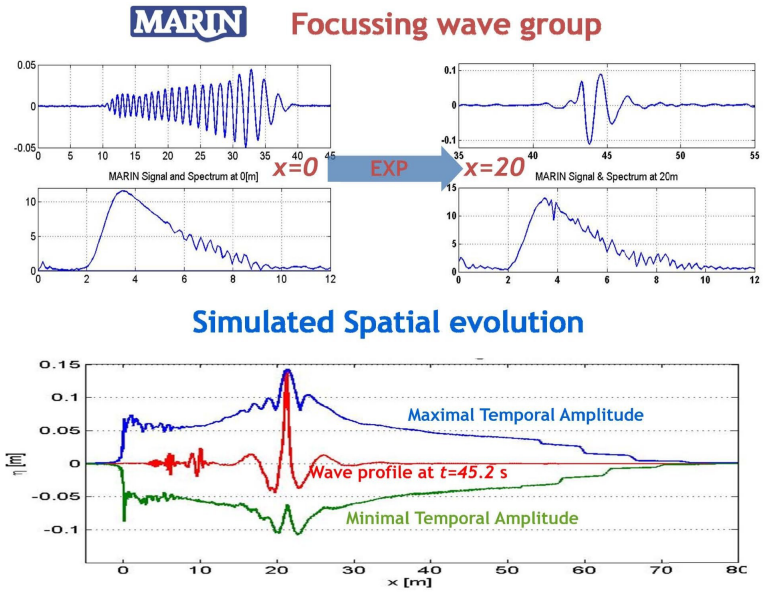


FIGURE 5. A mild focussing wavegroup, where faster running waves catch up with slower waves at a (designed) focussing position near 20m. In the lower plot elements of the propagation in de tank starting at 0 are shown by the curves of the Maximal and Minimal wave heights at each position.

wave length contributing to the initial profile, will travel fastest, waves with shorter wave lengths will travel slower. Hence (half of) the initial profile will become broader with increasing time.

This explains that waves that are originally generated by a storm in the ocean, and then travel and approach the coast, are 'separated ': the longest waves that will arrive first, called *swell*, followed by shorter waves (that usually have much less wave height and are less observable).

Dispersion is also used in a hydrodynamic laboratory to make a very high wave by *dispersive focussing*, waves that are much higher than could be done by the wave board in a single stroke. The effect of dispersion, and the (nearly) constancy of the spectrum for *linear* propagation of unidirectional waves, is well illustrated by the plots in Fig.5 of a focussing wave group generated and measured in a long wave tank at the hydrodynamic laboratory MARIN (Maritime Research Institute Netherlands).

At the wave flap (at the left of the tank, at $x = 0$), first higher frequency (shorter wave lengths) waves of moderate amplitude are generated, followed by smaller frequency (longer) higher waves, as is shown by the time signal at $x = 0$. While running to the right, the shorter waves travel slower than the longer waves, so that the longer waves catch up with the slower ones. Then superposition of the colliding waves gives a substantial amplitude amplification, here at a position approximately $x = 20m$ from the wave flap, as shown in the time signal at the right. This specific phenomenon happens only because the phases of the generated waves are taken very special at the wave flap. As we can see, the spectra of both signals are (almost) the same, indicating that this case is close to the linear evolution.

A total view of the dynamic evolution in the wave tank is shown at the lower plot which is obtained by a numerical simulation (with the AB-equation) since such measurements cannot be done in the wave tank. In that plot the maximal and minimal surface elevation at each position (the MTA: *Maximal and Minimal Temporal Amplitude*, respectively) is shown, together with a snapshot of the wave profile at the moment that the maximal height is obtained. Observe that according to this simulation the position of highest elevation is actually around $21m$, somewhat different from the targeted position; this illustrates that such simulations can be of help for laboratory purposes, for instance when the performance of a ship in high waves has to be investigated.

4. Long-crested Sea waves (irregular waves)

In reality the sea surface elevation consists of a combination of waves of different wavelength from different directions. We first concentrate on unidirectional waves running in the direction of the positive x-axis: long-crested waves. Later short crested waves in 2 horizontal dimensions will be described.

An almost perfect reference to the topics discussed (in much more detail than) here is the book

L.H. Holthuijsen, *Waves in Oceanic and Coastal Waters* 2007

4.1. Irregular waves. In the linear case of waves with sufficiently small amplitude, we can assume the elevation to be a pointwise superposition, so

$$\eta(x,t) = \Sigma_m a_m \cos(k_m x - \omega_m t + \phi_m) \text{ with } \omega_m = \Omega(k_m) \tag{1.28}$$

where the amplitudes a_m and the phases are not related to each other; the phases are modelled to be random with uniform distribution in $[0, 2\pi]$. Linear superposition of many wave modes leads to complicated wave patterns.

We will use the continuous description with Fourier integrals in the following very regularly. A superposition of to the right travelling waves is then described as

$$\eta(x,t) = \int_{-\infty}^{\infty} \alpha(k)\, e^{i(kx-\Omega(k)t)} dk \tag{1.29}$$

This is real valued provided $\alpha(k) = \overline{\alpha(-k)}$ where the bar denotes complex conjugation.

REMARK 1. *Note that if we write the complex amplitude as*

$$\alpha(k) = a(k)\, e^{i\theta(k)}$$

with $a(k)$ the real amplitude and $\theta(k)$ a real phase, the elevation η will be real only if the phase is skew symmetric:

$$\theta(-k) = -\theta(k)$$

The *initial profile* is given at $t = 0$ by

$$\eta(x,0) = \int \alpha(k)\, e^{i(kx)} dk$$

i.e. $\alpha(k)$ is the Fourier transform of $\eta(x,0)$

NOTATION 1. *Here and in the following we use notation for Fourier (inverse) transform in space and time as*

$$f(x) = \int \hat{f}(k)\, e^{ikx} dk, \quad \hat{f}(k) = \frac{1}{2\pi} \int f(x)\, e^{-ikx} dk$$

$$s(t) = \int \check{s}(\omega)\, e^{-i\omega t} d\omega, \quad \check{s}(\omega) = \frac{1}{2\pi} \int s(t)\, e^{i\omega t} d\omega$$

Likewise, the time signal at position $x = 0$ (as measured by a buoy there) is given by

$$\eta(0,t) = \int \alpha(k)\, e^{-i\Omega(k)t} dk$$

which can be rewritten by a transformation in the integral from k to ω. To that end we use the inverse of the dispersion relation (which exists since the dispersion relation for water waves is monotonically increasing) with the notation

$$\omega = \Omega(k) \leftrightarrow k = K(\omega) \tag{1.30}$$

(1.31)
$$\eta(0,t)=\int\alpha(k)\,e^{-i\Omega(k)t}\frac{dk}{d\omega}d\omega=\int\alpha(K(\omega))\,e^{-i\omega t}\frac{dK(\omega)}{d\omega}d\omega\equiv\int\sigma(\omega)\,e^{-i\omega t}d\omega$$

where now $\sigma(\omega)$ is the Fourier transform in time (temporal FT) of the *point signal* (at $x=0$ in this case). Observe that the Jacobian of the transformation is actually the inverse of the group velocity, expressed as function of ω

$$\frac{dK(\omega)}{d\omega}=\frac{1}{V_g(K(\omega))} \tag{1.32}$$

This result is of much importance for unidirectional influxing, as we shall see later; therefore we formulate the result explicitly.

COROLLARY 1. *A unidirectional irregular wave can be defined uniquely by prescribing the signal at a certain position, or the profile at a certain time. The relation between the signal of the wave height $s(t)$ at a certain position, say $x=0$, and the wave profile $p(x)$ at a certain time, say $t=0$, that lead to the same dynamic wave is given by*

$$\alpha(k)=V_g(k)\,\sigma(\Omega(k))\ \textit{for all } k \tag{1.33}$$

where $\alpha(k)$ is the spatial Fourier transform of $p(x)$ and $\sigma(\omega)$ is the temporal Fourier transform of $s(t)$.

4.2. Sea spectra. The variance of an irregular wave at a fixed position $x=0$ can be calculated, assuming the mean value is zero, as

$$var(\eta(0,t))=\lim_{T\to\infty}\frac{1}{T}\int_0^T\eta(0,t)^2\,dt=2\pi\int|\sigma(\omega)|^2\,d\omega \tag{1.34}$$

where $\sigma(\omega)$ is the Fourier transform of the time signal $\eta(0,t)$ at $x=0$

$$\eta(0,t)=\int\sigma(\omega)\,e^{-i\omega t}d\omega \tag{1.35}$$

This motivates to define the (two-sided) *energy spectrum* $E(\omega)$ such that

$$var(\eta(0,t))=\int E(\omega)\,d\omega\approx\Sigma_n E(\omega_n)\,\Delta\omega$$

Reversely, given the (discrete) energy spectrum $E(\omega)$, the amplitudes of the harmonic modes are determined by $a_m^2/2=E(\omega_m)\,\Delta\omega$, or by $|\sigma(\omega)|$ in the continuous description, i.e.

$$a_m=\sqrt{2E(\omega_m)\,\Delta\omega}$$
$$|\sigma(\omega)|=\sqrt{E(\omega)}$$

The complex amplitude $\sigma(\omega)$ has absolute value $|\sigma(\omega)|$ that is determined by the spectrum. A sea spectrum is specific for the sea: it provides info about the amount of energy with frequency ω, i.e. $|\sigma(\omega)|^2$, but has no phase information

$$S(\omega)=|\sigma(\omega)|^2$$

Although there are many different spectra that describe different types of sea, often used is the so-called *Jonswap-spectrum*, that has various parameters with which a variety of seas can be described; see below.

The width of the spectrum determines the 'irregularity' of the waves, related to the fact from Fourier theory that *a narrow spectrum corresponds with a broad spatial function, and, reversely*, a broad spectrum will produce a narrow spatial function[1]:

- a harmonic wave (which has infinite broadness) has only a single frequency and the spectrum resembles a delta-function, the limiting case of a very narrow spectrum,
- a narrow spectrum has the energy concentrated in a rather small frequency interval; the various wave contributions from frequencies in this interval produce modulations of the harmonic wave that corresponds with the frequency of some centre value, and the wave is mildly irregular;
- a combination of narrow spectrum peaks with different frequencies may lead to very high waves because of nonlinear effects; the linear case of a bi-harmonic is the simplest example;
- a broad spectrum makes the wave profile very irregular; the superposition leads to waves of various amplitude with rather high waves when there is an accidental constructive interference, to very small when destructive interference takes place.;
- an unrealistic, limiting case of a broad spectrum is one with E constant for all frequencies; if all phases vanish, the superposition would lead to a delta-function in physical space.

From constructive interference of waves in oceans with a broad spectrum, very high waves may emerge, so called *freak waves* that have a wave height exceeding twice the significant wave height; typically one or two of such waves will appear per 2000 waves. In seas with more confined spectra, such waves may also arise because of nonlinear effects (combined with constructive interference).

4.2.1. *Empirical wave spectra.* Waves in the ocean are generated by the wind, which is rather fluctuating in strength and direction. Roughly speaking the wind generates waves that will run approximately in the direction of the wind, and with wave height that depends on the wind: the stronger the wind, the higher and longer the waves. One distinguishes between *fully developed seas* for which an equilibrium is reached between the generation process and the wind strength, and *fetch-limited seas* where the generation process is still developing in time and space (fetch). Measurements of various sea states have revealed the typical spectra that correspond to these situations, with an explicit simple functional description containing parameters to tune to a specific situation. These spectra are briefly described next.

For *fully developed waves in deep water*, the *Pierson-Moskowitz spectrum* (one-sided) is described as

$$E_{PM}(\omega) = \alpha_0 g^2 \left(\frac{\omega_p}{\omega}\right)^5 \exp\left(-\frac{5}{4}\left(\frac{\omega_p}{\omega}\right)^4\right) \tag{1.36}$$

[1]This is most clearly illustrated by the Gaussian function $G(x)$ and its Fourier transform $\hat{G}(k)$ given by

$$G(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad \hat{G}(k) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{\sigma^2 k^2}{2}\right)$$

which shows the reversed dependence of the width of these functions determined by σ.

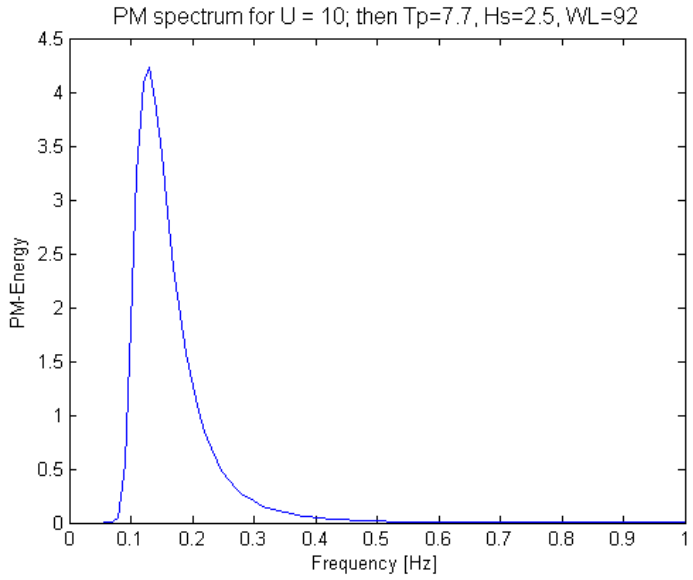


FIGURE 6. Example of a Pierson-Moskowitz spectrum, for the case of fully developed wind waves above deep water at a windpseed of $U = 10m/s$; the peak period is then $T_p = 7.7s$, the wavelength $L_p = 92m$, and the significant wave height $H_s = 2.5m$.

Here ω_p is the peak frequency and α_0 is an energy scale related to the total energy (variance) since by integrating over all frequencies one gets

$$\int E_{PM}(\omega)\, d\omega = \alpha_0 g^2/5$$

The main peculiarity of this spectrum is the decay-factor ω^{-5} for short waves, while the exponential provides a smooth transition to longer waves. This ω^{-5} behaviour includes dominant effects of wave breaking; a ω^{-4} tail is also used as an alternative (Donelan spectrum). These spectra are very broad, carrying waves with a large variation in frequencies.

For *fetch-limited waves in deep water*, a modification of the PM spectrum has been found, the so-called *JONSWAP spectrum*, which allows for a much higher and more narrow shape around the peak frequency:

$$E_{JS}(\omega) = E_{PM}(\omega).G(\omega) \tag{1.37}$$

$$\text{with } G(\omega) = \gamma^q \text{ with } q = \exp\left[-\frac{1}{2}\left(\frac{\omega/\omega_p - 1}{\sigma}\right)^2\right] \tag{1.38}$$

The quantity γ is the peak-enhancement factor, σ is a peak-width parameter with different values before and after the peak frequency ω_p

$$\sigma = \begin{cases} \sigma_a & \text{for} \quad \omega < \omega_p \\ \sigma_b & \text{for} \quad \omega > \omega_p \end{cases} \tag{1.39}$$

Typical values are $\gamma \in [1, 10]$ (larger values produce higher and more confined spectra), $\sigma_a = 0.07, \sigma_b = 0.09$.

For increased time (and fetch), a JS-spectrum will develop into a PM spectrum mainly by nonlinear wave interactions. The JONSWAP spectrum is most often used.

4.2.2. *Wave properties from spectra.* By assuming that the random sea-surface is a *stationary Gaussian process* (which is implied by the superposition of many independent waves of different frequency and random phase), the energy spectrum $E(\omega)$ determines all the statistical characteristics of the wave field.

To describe the most important ones, we use the definition of the m^{th} moments:

$$m_n = \int \omega^n E(\omega)\, d\omega \tag{1.40}$$

- The **total Energy**, which is the variance of the elevation, is given by :

$$m_0 = \int E(\omega)\, d\omega \tag{1.41}$$

- **Wave period and frequency**. Because of different possible definitions of a 'wave period' there are various alternatives.
  For the mean zero-crossing period $\bar{T}_0$ one gets

$$\bar{T}_0 = \sqrt{m_0/m_2} \tag{1.42}$$

  Another possibility is to take the mean frequency of the spectrum, denoted by

$$T_{m01} = \frac{2\pi}{\omega_{mean}} = \frac{m_0}{m_1}$$

  The **peak frequency** ω_p is the frequency of maximal energy (the position of the 'peak'):

$$\max_{\omega} E(\omega) = E(\omega_{peak}), \text{ and } T_{peak} = 2\pi/\omega_{peak} \tag{1.43}$$

  This is approximately the same as the period of the $1/3^{th}$ longest periods in a wave record.
- **Significant wave height** H_s is, together with the peak frequency, the most used term in describing sea and ocean waves. Originally used to indicate the average wave height as estimated by experienced observers of waves in natural surroundings, it turned out that this estimated value was actually much closer to a different description, which is now the current definition, as

  $H_s =$ *averaged wave height of the* $1/3^{th}$ *highest waves in a record*

  Using the spectrum the significant wave height can be well approximated by

$$H_s \approx 4\sqrt{m_0} \textit{ in deep water} \tag{1.44}$$

  To indicate that this approximation is used, one may write H_{mo} instead of H_s.
- **Maximum wave height.** The largest wave height $H_{\max}$ in a record of N waves can be shown to be approximated by

$$H_{\max} \approx 2\sqrt{m_0}\sqrt{2\ln N} \tag{1.45}$$

  For a record with $N \in [1000, 4000]$ it holds $\sqrt{2\ln N} \in [3.72, 4.07]$ and hence to a good degree we get

$$H_{\max} \approx 2H_s \text{ for } N \in [1000, 4000]$$

A typical storm with waves of $T_p = 10s$, carries 360 waves per hour; *if the duration of a storm is 6 hours or more, we can expect that* at most positions *(!) a wave with height* $2H_s$ *will have passed.*

4.3. Randomness and consequences. For given spectrum, the unidirectional sea states can be described for given spectrum as

$$\eta(x,t) = \int_{-\infty}^{\infty} \sqrt{S(\omega)} e^{i(K(\omega)x - \omega t + \theta(\omega))} d\omega \tag{1.46}$$

Here the *random phase* $\theta(\omega)$ (satisfying $\theta(-\omega) = -\theta(\omega)$ to make sure the elevation is real valued) turns up, which is assumed to be a random (stochastic) variable: it is *uniformly distributed* (=arbitrarily chosen) in the interval $[0, 2\pi)$.

The role of the random phase is essential. On the one hand it reflects the 'unpredictable' nature of the sea because of several physical perturbing causes that cannot be described deterministically; it expresses the 'arbitrariness ' of the sea waves, perturbed by wind, bottom variations, etc.

The importance of this random phase can be appreciated by observing that if at a certain time and place all phases would vanish, say at $x = 0, t = 0$, all wave components would have their highest amplitude there (see the example of a focussing wave above; for irregular waves this it was called a *maximal wave*). A consequence of this *constructive interference* would be that the resulting crest height at that exceptional point and time is given by

$$\int \sqrt{S(\omega)} d\omega$$

which is an order of magnitude larger than realistic.

The random phases prevent this to happen (that is, will have probability zero), and leads to a sea that has the following *statistical properties* which are believed to be realistic when the depth is deep (except for very high seas during storms when nonlinear effects disturb these properties somewhat):

- the sea is *uniform*: although at each instant there are many differences between neighbouring points, when observed over a long time, all the characteristic quantities (such as H_s) are the same at each point
- the sea is *ergodic*: measuring the sea over a long time, the same results are obtained as if the sea has been measured during many different observations
- the elevation is statistically given by a *Gaussian elevation distribution* (at each position) as a consequence of the central limit theorem of statistics.

5. Empirical Wind-Wave relations for deep water

Most waves in the ocean and seas are generated by wind, i.e. by atmospheric pressure differences that exerts different forces in time and space on an initially flat water surface. Once the water level is disturbed, the gravity will cause the elevation to deform in time. Since dissipation in water is negligible (except during breaking and bottom friction near the coasts), the energy of the disturbance remains practically constant but will be spread around: 'waves' will start to travel away from the disturbance.

If a wind is blowing with sufficient strength, various effects will cause the waves to grow until a certain maximal wave height is obtained, the height depending on the wind strength. The propagation will be approximately in the direction of the wind, and the speed of the waves will be close to that of the speed of the wind.

The precise physical processes of wave generation by winds are quite complicated, and are difficult to model; the results depend mainly on depth, wind duration, and sufficient distance (fetch) for the wave to travel before reaching land areas. Yet for one special case, there are simple empirical formula that describe the main properties of the generated (irregular) waves solely by the strength of the wind. That is the case for so-called *fully developed deep water waves*, meaning that the waves had sufficient time and distance to travel to be built up to the point that they are in an equilibrium situation for a given wind strength. Below we present these empirical algebraic relations. They can be used as approximations to quickly estimate wave properties for given wind. When the conditions are not satisfied (for instance for waves above varying bathymetry), numerical programmes like SWAN have to be used to calculate statistical properties of the resulting waves from given wind data, or phase-resolved codes that simulate in time full details of the waves.

In the following we denote by U the *wind strength at 10 m height above the water surface*. Then with only U and g as determining quantities, the following relations can be found from *dimension analysis*, but the value of the coefficients for H_s and T_p have been found empirically by matching with observations:

- **Significant wave height**

$$H_s = 0.24\frac{U^2}{g} \tag{1.47}$$

- **Peak period and peak frequency**

$$T_p = 7.69\frac{U}{g},\ \omega_p = 0.82\frac{g}{U} \tag{1.48}$$

With these results from observations, we find the derived quantities using the dispersion relation:

- **Wave length** (found from deep water dispersion and T_p above):

$$\lambda_p = 9.410\frac{U^2}{g} \tag{1.49}$$

- **Phase speed** can be derived from $C = \lambda_p/T$ as

$$C = 1.22U \tag{1.50}$$

- **Steepness** follows with the quantities above:

$$ak = \pi\frac{H_s}{\lambda} = 0.08 \tag{1.51}$$

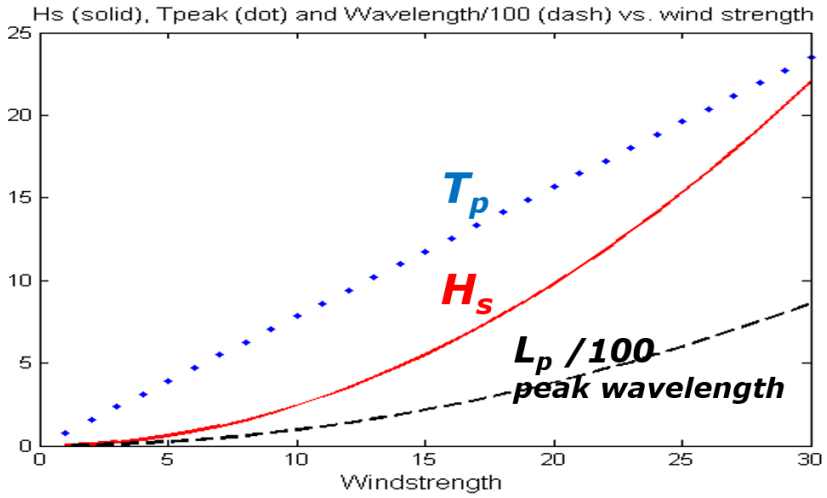


FIGURE 7. As function of the windstrength are shown: The significant waveheight (solid, middle curve), the peak period (dotted, upper line), and wavelength/100 (dashed, lower parabola).

- **Pierson-Moskowitz Spectrum**. Given H_s and T_p as above, the spectrum of the irregular wind waves described by the Pierson-Moskowitz spectrum is completely determined

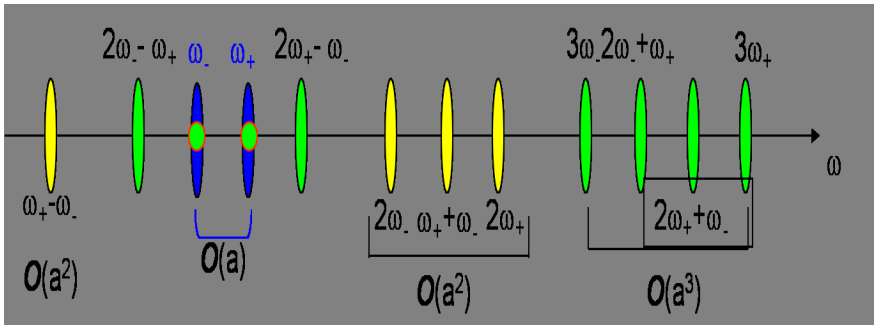


FIGURE 8. Illustration of mode interaction

6. Nonlinearity

In the previous subsections we used abundantly the assumption that the wave model was assumed to be linear. Because of that assumption, superpositions of solutions are solutions again, which we used in the description with the Fourier integrals. This assumption of linearity is an approximation that is accurate only if the surface elevation is sufficiently small, small compared to the depth of the layer and compared to the wave length. This last condition means that the waves should not be too 'steep'. For higher (steeper) waves, nonlinear effects become more important and can lead to large disturbances, both in amplitude as well as in propagation speed. A superposition of solutions will not be a solution, so that expansion techniques (like Fourier methods) are not valid for nonlinear equations.

For water waves the nonlinearity is quadratic, as we shall see. It would be wrong to think that this is a simple nonlinearity: the effect of nonlinearity cannot simply be classified by the order, nor that the effects are easily categorized by that. This is already quite familiar for solutions of the simple ode $\dot{x} = x^n$, which shows the essential different behaviour for the linear case $n = 1$ and the nonlinear cases with $n < 1$ and $n > 1$.

6.1. Mode interaction. For linear equations we have seen that superposition is a powerful tool; it makes it possible to write down the general solution as superposition of harmonic modes. One reason that this is possible is that the energy of each mode is conserved. However, *for nonlinear equations modes will exchange energy between each other* because the mode equations are coupled.

This can be illustrated by the fact that the Fourier transform of a product becomes the convolution of the Fourier transforms. This is illustrated in Fig.8. Two original modes of amplitude order a, and frequencies $\omega_\pm$ produce higher order modes by quadratic nonlinearity: second order modes like $\omega_+ \pm \omega_\pm$ that will have amplitude of order a^2, and third order modes have amplitude a^3, and so on.

This mode generation and interaction causes energy exchange, as we see in the spectrum of the Strong Focussing Wave Group in Fig.9.

6.2. Breaking wave equation. As an example of possible consequence of nonlinearity, we perturb the translation equation: instead of a constant velocity, we add to the velocity a (possibly small) fraction a of the solution:

$$\partial_t \eta = -\left(c_0 + a\eta\right) \partial_x \eta \tag{1.52}$$

This means that larger amplitude modes travel somewhat faster (if $a > 0$) then smaller amplitude modes. For this simple case the solution can even be written down (which is exceptional for nonlinear equations), albeit in an implicit way. Indeed, by observing that

$$\frac{d}{dt}\eta\left(x\left(t\right),t\right)=0 \text{ for } \frac{dx}{dt}=c_0+a\eta \tag{1.53}$$

we find that η is actually constant on the characteristics (lines in the x,t-plane given by the solutions of $dx/dt = c_0 + a\eta$). Hence

$$\eta\left(x,t\right)=f\left(\xi\right) \text{ for } x-\left[c_0+af\left(\xi\right)\right]t=\xi \tag{1.54}$$

The solution for increasing time will become a multi-valued function of x for most initial functions. This resembles somewhat the breaking of a wave, which explains the name for the equation.

If we try to find the solution in the form of a Fourier series, say $\eta\left(x,t\right)=\Sigma_k\hat{\eta}_k\left(t\right)e^{ikx}$, we get the set of nonlinear coupled equations for the coefficients $\hat{\eta}_k$:

$$\partial_t\hat{\eta}_k=-c_0\left(ik\right)\hat{\eta}_k-a\Sigma_{m+\ell=k}\left(im\right)\hat{\eta}_m\hat{\eta}_\ell$$

The quadratic terms turn up because modes e^{imx} and $e^{i\ell x}$ contribute to mode e^{ikx} if $m+\ell=k$, as illustrated above for 2 modes. Hence the nonlinearity causes *mode generation and mode coupling.*

We will not go into further details here, but will give some illustrations of the effect of nonlinearity in cases with rather simple wave forms.

6.3. Nonlinear focussing waves.

6.3.1. *Focussing of a confined wave.* A more extreme case of a focussing wave than the one shown before, is illustrated with another MARIN experiment that shows nonlinear effects. Although the focussing phenomenon is mainly caused by dispersive effects, this case shows subtle, but relevant nonlinear effects. This focussing case is shown in Fig. 9, starting from $10m$ and focussing at $50m$, with wave forms and their spectra at these positions, but also at (non-equidistant!) positions in between. This experiment at MARIN is an extreme case in the sense that observations during the experiment reported some 'white capping' (a mild form of breaking) near the focal point. The simulation reveals almost perfectly the measured elevation. The changes in the spectra from one position to another also show the substantial effect of nonlinearity; during the evolution in the tank, and most clearly close to the focussing point, long waves and short waves are generated.

6.3.2. *Soliton on Finite Background.* One other example of the effect of nonlinearity is shown in Fig. 10; this well-structured phenomenon is caused by a rather sensitive combination of the nonlinear mode coupling as described above and dispersive effects.

The phenomenon can be seen as the process that starts at the left in Fig. 10 , where an harmonic wave train enters the domain (travelling to the right). However, the wave train is slightly perturbed: the amplitude is not constant but modulated somewhat. Provided this modulation is long enough, the wave train becomes 'unstable', the waves start to grow. This is the so-called *Benjamin-Feir* or *modulation instability.* The growth is caused by nonlinear interactions, that change the wavelength slightly while propagating. This changing wavelength causes an increase of amplitude. This leads initially to an exponential increase of the amplitude, but nonlinearity controls the growth. After a certain maximal wave height is obtained, the reverse process sets in: the wave height decreases again, and leads to a harmonic wave train further down stream.

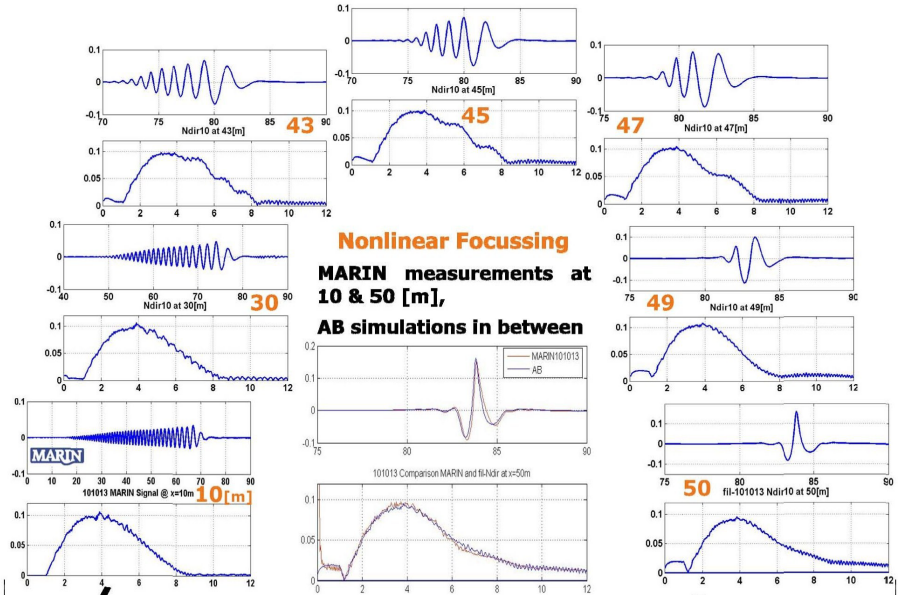


FIGURE 9. Strong focussing wavegroup, with substantial nonlinear effects as seen by the long and short wave generation near the focussing point at 50m.

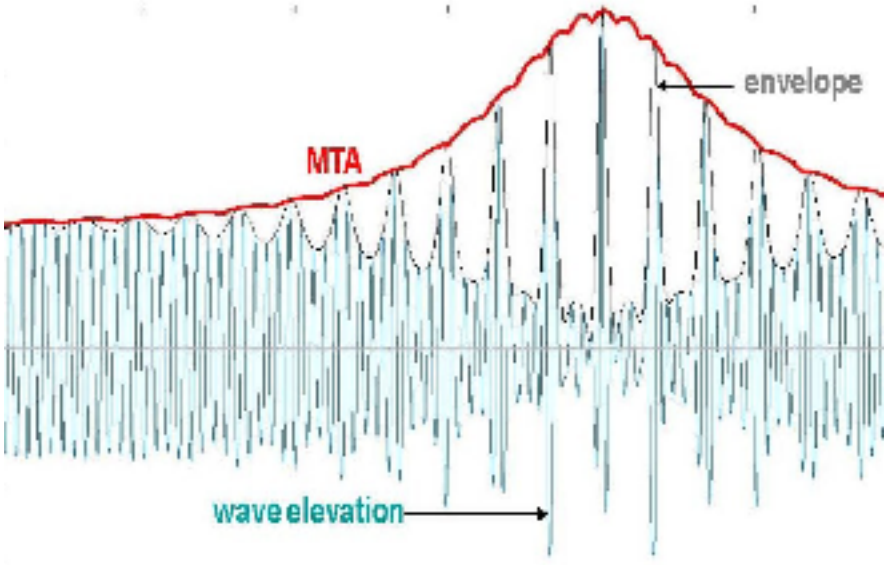


FIGURE 10. Plot of the Soliton on Finite Background, horizontally the distance and vertically the wave elevation. Shown is a snapshot of the individual waves, the envelope that is formed by groups of a few waves, and the MTA (Maximal Temporal Amplitude). While the waves and the envelope will move in time, the MTA is steady.

This full nonlinear evolution can actually be described explicitly in closed formulas. It is called the *Soliton on Finite Background*; the word 'soliton' refers to the shape of the steady envelope of the wave train, the MTA as shown in the plot. The word 'background' refers to the harmonic wave train. The fact that an explicit description is available makes this case very remarkable and very exceptional for nonlinear problems.

Nowadays it is believed that this modulation phenomenon may also play an important role in the appearance of *freak waves*: extremely high waves that all of a sudden can appear on seas and oceans. Most of the waves will be of only moderate height, but the freak wave will be at least twice as large. Since such waves (also called rogue waves, or giant waves) can seriously damage ships, regular warnings are issued for oceanic traffic; one of the criteria used for prediction is the so-called Benjamin-Feir index, related to the ratio of wavelength of the waves and length of the modulation.

6.4. Breaking. Breaking of waves is a quite familiar phenomenon. It can be observed when walking along the beach where breaking seems to be restricted to the near beach area. The shoaling effect will make waves higher, seemingly until they are too high or too steep, after which they will break. But also on deep water wave breaking takes place, visible from an airplane when flying to an airport close to the beach. The form in which wave breaking can occur is usually described as spilling, splashing or overturning breaking.

The full details of the breaking process and the initiation are not yet well understood. But it is understood that *during breaking energy is dissipated*, but that the *horizontal momentum is conserved*. As breaking criterion a kinematic condition can be used: breaking will be initiated when the quotient of particle velocity at the crest (U_c) and the actual velocity of the crest (V_c) exceeds a certain value: $U_c/V_c > crit$, with $crit$ a value somewhat less than 1; the precise value cannot be calculated precisely (yet) but has to be found based on empirical data.

Fig.11 shows a result of a simulation of run-up on a coast; the circles (resembling white capping) indicate the breaking positions where the kinematic breaking criterion is satisfied.

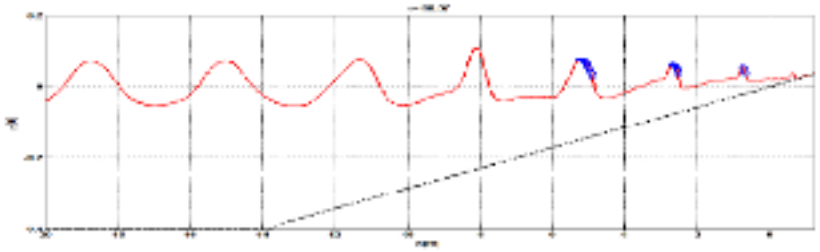


FIGURE 11. Run-up on a sloping coast; breaking is indicated by the circles; amplitide decrease caused by the breaking is clearly observable.

Fig. 12 illustrates the large impact of energy dissipation caused by breaking. Shown are time traces of simulations at a position downstream of an under water bar in a laboratory that caused the waves to break heavily; the simulation without breaking (red dashed) shows very high wave (near $t = 290, 440$) while the simulation with breaking (dotted black), which nearly coincides with the measurement (solid blue), has much reduced wave height there.

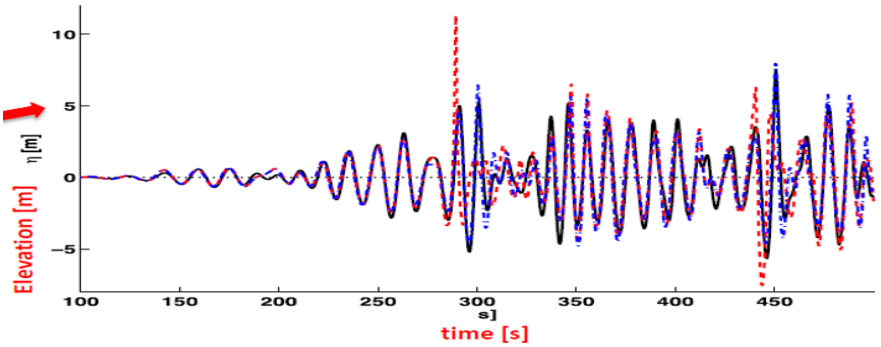


FIGURE 12. Illustration of the effect of breaking: the simulation without breaking mechanism (dashed line) shows unrealistic high wave elevation compared to the simulation with breaking at a position after breaking occurred.

7. Short-crested 2HD waves

Long crested waves as treated above are very different from the actual waves in the sea. The generalization from 1HD to 2HD spatial horizontal directions is formally rather direct. A major role is now being played by a multi-directional spectrum $E(\omega, \theta)$, which provides the spectrum in each direction θ. But there are still serious questions what type of spectra do actually describe realistic sea surfaces; measurements in 2HD are much less available than in 1HD due to the more problematic way to produce and interpret them.

7.1. Multi-directional sea waves. For 2HD we use the following notation: for the position in (x, y) space $\mathbf{x} = (x, y)$ and for the *wave vector* $\mathbf{k}$

$$\mathbf{k} = (k_x, k_y) = k(\cos\theta, \sin\theta), \text{ with } k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2}$$

where k is the length of the wave vector $\mathbf{k}$. A single harmonic mode is then given by

$$\eta(x,t) = a\cos(\mathbf{k}\cdot\mathbf{x} - \omega t + \phi) = a\cos(k_x x + k_y y - \omega t + \phi) \tag{1.55}$$

For the dispersion relation, the same relations as in 1HD holds, with $k = |\mathbf{k}|$, but we have to distinguish the direction the wave travels. To specify that, take any direction $\mathbf{e}$, and call a wave *forward propagating* with respect to $\mathbf{e}$ if the phase speed is directed in the half-plane that has $\mathbf{k}$ as normal direction. Then the dispersion relation reads

$$\Omega_2(\mathbf{k}) = \text{sign}(\mathbf{e}\cdot\mathbf{k})\,\Omega(|\mathbf{k}|) \tag{1.56}$$

To understand the propagation direction, take $\mathbf{e} = \mathbf{e}_x$ the direction in the positive x-direction; then $k_x > 0$ is the half plane of positive directions. Stated differently, wave modes

$$\exp i(\mathbf{k}\cdot\mathbf{x} - \omega t) = \exp i\left(k_x\left(x - \frac{\omega}{k_x}t\right) + k_y y\right)$$

with $k_x > 0$ will be running to the right into the positive half-plane $x > 0$, and with $k_x < 0$ into the negative half-plane.

Irregular waves are formed as superposition of various modes:

$$\eta(\mathbf{x},t) = \Sigma_{n,m} a_{mn}\cos(\mathbf{k}_{nm}\cdot\mathbf{x} - \Omega_2(\mathbf{k}_{nm})\,t + \phi_{nm}) \tag{1.57}$$

The summation is over two indices (double summation $\Sigma_{n,m} = \Sigma_n \Sigma_m$), related to the two components of a wave vector: $\mathbf{k}_{nm} = k(\omega_n)(\cos\theta_m, \sin\theta_m)$. Written with Fourier integrals and complex amplitudes

$$\eta(\mathbf{x}, t) = \iint \alpha(m, n) \exp i(\mathbf{k}_{nm} \cdot \mathbf{x} - \Omega_2(\mathbf{k})\, t)\, dmdn$$

The variance at a fixed point $\mathbf{x}_0$ is defined as before given by

$$var(\eta(\mathbf{x}_0, t)) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \eta(\mathbf{x}_0, t)^2\, dt \tag{1.58}$$

This motivates to define the *directional energy spectrum* $E(\omega, \theta)$, also called the *frequency-direction spectrum* or two-dimensional spectrum, such that

$$var(\eta(\mathbf{x}_0, t)) = \int E(\omega, \theta)\, d\omega \tag{1.59}$$

Reversely, given a (discrete, one-sided) energy spectrum $E(\omega, \theta)$, the amplitudes of the harmonic modes are determined by $a_{mn}^2/2 = E(\omega_{mn}, \theta_{mn})\, \Delta\omega\Delta\theta$, i.e.

$$a_{mn} = \sqrt{2E(\omega_{mn}, \theta_{mn})\, \Delta\omega\Delta\theta} \tag{1.60}$$

REMARK 2. *As in 1HD, the amplitudes a_{mn} can be obtained from the information of an initial wave profile $\eta_0(\mathbf{x}) = \eta(\mathbf{x}, t_0)$. But now the information of the elevation signal at a single point $\eta(\mathbf{x}_0, t)$ is* ***not*** *sufficient (since this carries only 1D information, while 2D information is needed); information about the elevation on a curve, for instance the y-axis, will be sufficient: $s(y, t) = \eta(0, y, t)$.*

It is with present state of knowledge not very clear which directional energy spectra describe actual sea states. Measurements of surface elevations at sea are usually restricted to point measurements, where the measured wave height is the result of the collective contribution of waves from all directions. Hence, the experimentally observed spectra that can be described as a PM of JS-spectrum, say $E_1(\omega)$, are actually the direction integrated results of directional spectra:

$$E_1(\omega) = \int_{-\pi}^{\pi} E(\omega, \theta)\, d\theta; \tag{1.61}$$

these are therefore called the *one-dimensional spectra*. Writing

$$E(\omega, \theta) = E_1(\omega)\, D(\omega, \theta) \tag{1.62}$$

we introduce the *directional spreading function* $D(\omega, \theta)$ which is normalized as

$$\int D(\omega, \theta)\, d\theta = 1. \tag{1.63}$$

Observe that long crested waves travelling in a direction θ_0 are described formally with $D(\omega, \theta) = \delta_{Dirac}(\theta - \theta_0)$.

Realistic sea waves are usually described with the help of a spreading function given by a $\cos^{2s}$-function:

$$D(\theta; s) = A \cos^{2s}((\theta - \theta_0)/2) \text{ for } |\theta - \theta_0| < \pi \tag{1.64}$$

with the normalization coefficient A taken suitably.

REMARK 3. *In various publications this spreading function is taken without the factor $\frac{1}{2}$ in the argument of the cosine; moreover, most times, the spreading is restricted to forward propagating waves (with respect to the direction θ_0).*

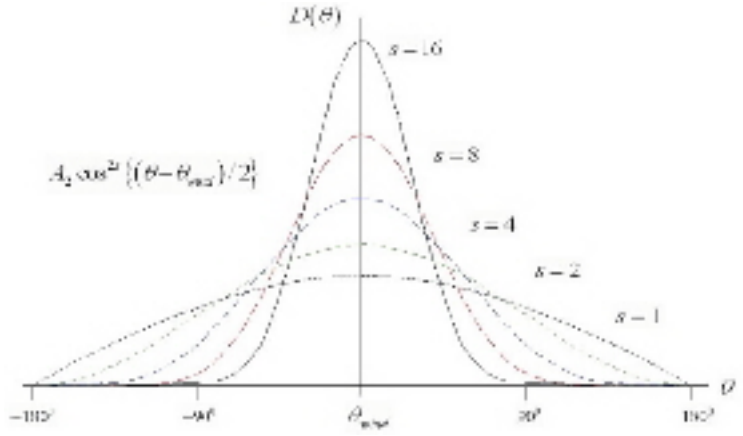


FIGURE 13. Spreading function for multi-directional waves.

For wind waves there is an empirical relation between the power s and the wind strength. Then the spreading function depends also on the frequency through the power s, namely

$$s = 2\sigma_\theta^{-2} - 1$$

where σ_θ is the standard deviation

$$\sigma_\theta = \int_{-\pi}^{\pi} \theta^2 D(\theta)\, d\theta$$

and empirically

$$\sigma_0 = 26.9 \left(\frac{\omega}{\omega_{peak}}\right)^{\alpha} \text{ with } \alpha = \begin{cases} -1.05 \text{ for } \omega < \omega_{peak} \\ 0.68 \text{ for } \omega > \omega_{peak} \end{cases}$$

REMARK 4. *Actually there is a serious problem with the 'double summation' description of irregular waves as in (1.57). Because of the randomness of the sea, we want a description that produces a so-called spatially homogeneous and time stationary wave field, roughly saying that wave properties should be* on average *the same at each position and at each time. And that is not the case for (1.57), because of the fact that more than one wave direction can correspond to the same frequency, as was noted by Jefferys 1987. If this happens, this will lead to wave interaction that form regular patterns. A simple example is the sum of two waves with the same frequency, one travelling in the x-direction, the other in the y-direction; with the same amplitude and vanishing phases (for ease of presentation), we get with a simple goniometric formula*

$$\begin{aligned} &\cos(kx - \omega t) + \cos(ky - \omega t) \\ &= 2\cos(k(x+y)/2 - \omega t)\cos(k(x-y)/2) \end{aligned}$$

leading to a fixed modulation pattern $\cos(k(x-y)/2)$ *determined by the lines* $x - y = \xi$. *A practical way out is to restrict the description to a 'single summation' for which with each* ω *only one angle* θ *is associated:*

$$\eta = \Sigma_n a_n \cos(\mathbf{k}_n \cdot \mathbf{x} - \omega_n t + \theta_n) \tag{1.65}$$

There are various ways to do this; one choice is to choose for each ω *the value of* θ *randomly, but satisfying the probability determined by the directional spreading function.*

7.2. Special phenomena from Huygens' principle. Huygens' principle (1690) deals with the propagation of *wave fronts* in media such as light in air but also (in a good approximation) surface waves in water.

The basic idea is very simple, but the consequences lead to important phenomena like diffraction and refraction; a basic understanding is useful to interpret results of simulations.

Roughly speaking, Huygens' principle consists of two ideas:

(i) Inside a uniform medium, a point source (the primary source) will generate waves spreading in all directions forming a wave front with the shape of an expanding circle; in a non-uniform media the propagation speed will depend on the direction of propagation and the circle will be distorted;

(ii) The propagation of the front is described as if each point at the front acts like a (secondary) source point from which wave fronts are generated; the wave front at a later time is then the envelope of the wave fronts of all secondary sources at the original wave front.

Contemplating on the idea, it explains in a simple way the qualitative behaviour of diffraction and refraction, as described below[2].

Diffraction:

Waves above a flat bottom that meet a wall that partly blocks the waves, will partly invade the area behind the wall: there is no sharp transition but points at the wave front that were not blocked will produce secondary fronts that invade the area behind the wall, see Fig.14. Note that the edge of the wall is a singularity.

Refraction:

Waves above bathymetry will have propagation speed depending on the depth. This causes wave fronts to bend in such a way that the front propagation is perpendicular to the level lines of the bathymetry, see Fig.15. It explains why waves approach the shore line always (nearly) perpendicular, no matter the shape of the shore line.

[2]To be in line with Huygens' principle (the weak form of it, as in odd-dimensional cases), the following holds in the limit of shallow water waves for which waves of all wave length propagate with the same speed. For highly dispersive waves, the longest waves make up the 'front' which then becomes more vague.

FIGURE 14. Illustration of diffracting waves behind the breakwater.

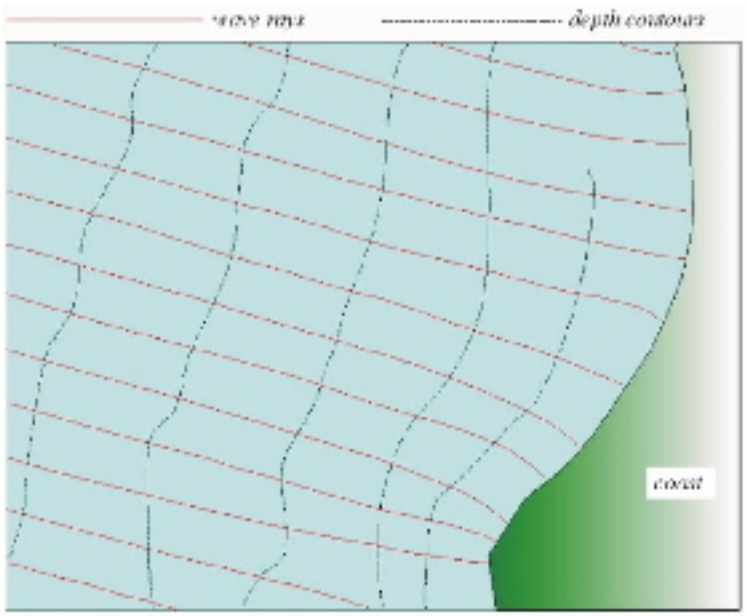


FIGURE 15. Illustration of refracting waves near the coast.

8. Simulations

It is suggested to download HAWASSI-Demo version(s) and perform simulations. Test cases are provided which illustrate many of the topics described in this chapter.

The following suggestions may serve as a guideline.

(1) Make simulations of 1HD linear harmonic waves; check that waves travel with the phase speed. Observe the entrance effect, which is described by an Airy function. Compare results of linear and nonlinear simulations, and observe differences (wave profile, crest height + trough-height, propagation speed).
(2) Make a simulation of a linear *bi-harmonic wave*; check the property that the carrier wave travels with the group speed. (Does the carrier wave satisfy the dispersion relation? And the envelope?) For reasonable values of the parameters, study non-linear bi-harmoinc waves and see (large) distortions.
(3) *1HD focussing wave*: using linear simulations, construct an influx signal that makes, at a specified position downstream, your own focussing wave that has a Gaussian form with amplitude 1.
(4) Perform a (long-time) simulation for an irregular wave. From the MTA of a snapshot, detect the position(s) of maximal height; then find the corresponding time signals at these positions; is there a Freak wave?. Then observe (animation) that the MTA in a neighbourhood of the maximal height is also determined by that high wave.
(5) Investigate 2HD harmonic waves, linearly and nonlinearly.
(6) Make a 2HD *oblique bi-harmonic wave*, influxing waves from one line in two different directions. Can you specify a direction that you would call a 'propagation' direction? Contemplate about the difficulty to define a 'propagation', and an 'energy' direction.
(7) *Scaling*
   (a) Suppose we perform an experiment on wave propagation at the moon in a similar set-up as on earth; will we find the same results/behaviour?
   (b) For an experiment in a wave tank of 5m depth, we want to simulate waves in nature which have peak-period 8s in water of depth 50m. Derive the following scaling laws for a scaling factor α:

$$\begin{cases} h \to \alpha h \\ x \to \alpha x \\ t \to \sqrt{\alpha} t \end{cases} \tag{1.66}$$

   (c) Performing simulations on the scale of the real waves and on the scaled experimental waves, will there be a difference in computation (cpu) time? And in relative computation time (=cpu time /time length of influx signal)?

CHAPTER 2

Dynamics of free surface water waves

In this chapter we will derive (summarize) the basic laws that govern the time evolution of water waves. It will be shown that the change of the free surface and the internal fluid motion are coupled by two dynamic equations at the surface. One kinematic equation (which is nothing but a 'mathematical law', as a consequence of the description of a continuum), and one dynamic equation that is essentially Newton's law but applied to continua as formulated by Euler. We will assume that water is an ideal fluid (which it is to a remarkably degree for water at constant temperature): it is *inviscid and incompressible*. We will normalize the constant mass density to be $\rho = 1$ and will suppress its appearance in the formulas below. (Any constant value will be scaled away, but this is not true anymore when ships are included.)

An *assumption* about the interior flow that is common when dealing with surface waves, is that the flow is *irrotational*. Then it can be described with a scalar potential Φ, and it will become possible to express the dynamic surface equation as the vanishing of the Bernoulli pressure. The potential is related to the three components of the velocity as

$$\mathbf{U}(x,y,t) = (U,V,W) = \nabla\Phi(x,y,t) \equiv (\partial_x\Phi, \partial_y\Phi, \partial_z\Phi) \tag{2.1}$$

with (x,y) the horizontal variables, and z the vertical coordinate. In the following we will regularly only use a single horizontal coordinate x to simplify the notation.

1. Bottom and lateral boundary conditions

The fluid domain will be bounded in the vertical direction by the given bottom profile, $z = -D(x)$, and the fluid surface $z = \eta(x,t)$. The bottom is taken to be impermeable, i.e. there is no water flow into the bottom. Denoting by

$$\mathbf{n}_B = (\partial_z D, 1)/\sqrt{1+(\partial_z D)^2}$$

the normal at the bottom, this gives the condition

$$\mathbf{U}\cdot\mathbf{n}_B = 0 \text{ at } z = -D$$

which is equivalent to the vanishing of the normal derivative of the potential

$$\partial_{\mathbf{n}_B}\Phi \equiv \nabla\Phi\cdot\mathbf{n}_B = 0 \text{ at } z = -D \tag{2.2}$$

In the horizontal direction, most times we reason as if the vertical lateral boundaries are far away from the interval in which we study the waves, and that velocities vanish there

$$\mathbf{U}\cdot\mathbf{n}_{lat} = \mathbf{U}\cdot\mathbf{e}_1 = (\partial_x\Phi, \partial_y\Phi)\cdot\mathbf{e}_1 = \partial_x\Phi \to 0 \text{ for } |(x,y)| \to \infty \tag{2.3}$$

For bounded domains, so for all simulations of waves in numerical codes, the properties at the (vertical) boundaries have to be prescribed, for instance for a hard wall we get

$$\textit{hard wall}\colon\ \mathbf{U}\cdot\mathbf{n} = 0$$

The opposite case, namely a *fully transparent boundary* is harder to formulate.

2. Interior potential flow, Airy theory

As stated in the introduction to this chapter, we assume that water is incompressible, which implies that the interior fluid velocity is divergence free:

$$\text{div}\mathbf{U} \equiv \nabla \cdot \mathbf{U} = 0 \tag{2.4}$$

The assumption that the flow is irrotational means that the rotation of the velocity $\mathbf{U}$ vanishes; this then implies that (and is implied if) $\mathbf{U}$ is a conservative vector field: there exist a scalar function, the *potential* Φ, so that $\mathbf{U}$ is the gradient of Φ

$$\text{curl}\mathbf{U} \equiv \nabla \times \mathbf{U} = 0 \Longleftrightarrow \mathbf{U} = \nabla\Phi \tag{2.5}$$

Instead of the 2 (or 3, in 3dimensions) components of the velocity field, now only one scalar Φ has to be dealt with, a substantial simplification.

The expression $\text{div}(\nabla\Phi) = 0$ can be written as

$$\text{div}(\nabla\Phi) \equiv \Delta\Phi \equiv \left(\partial_x^2 + \partial_z^2\right)\Phi = 0 \text{ in the interior} \tag{2.6}$$

which means we will deal with so-called *potential flows*.

> Related to the word *potential flows*, we will use the word '*potential*' in general for any interior flow that satisfies the Laplace equation in the interior and at the bottom and lateral boundaries the vanishing of the normal derivative.

2.1. Airy theory of potential flows. For arbitrary fluid domain, i.e. varying bottom and/or non-flat upper surface, explicit expressions for potential flows cannot be found. For an infinitely long strip, however, the potentials can be written down explicitly, as shown by Airy[1]. Physically a strip geometry means that the bottom is flat and that we assume that disturbance of the upper surface by waves can be neglected. Although this may seem uninteresting, it is the basis of the generalizations that are used in the HAWASSI software.

The idea is the simple method of separation of variables: for $\Phi(x,z)$ we look for solutions that can be described as

$$\Phi(x,z) = a(x)\, f(z)$$

Inserting in the Laplace equation one gets

$$\Delta\Phi = \partial_x^2\Phi + \partial_z^2\Phi = a''f + af'' = af\left(\frac{a''}{a} + \frac{f''}{f}\right)$$

which can only satisfy $\Delta\Phi = 0$ for all (x,z) provided

$$\frac{a''(x)}{a(x)} = -\frac{f''(z)}{f(z)} = -k^2 \text{ (a constant)}$$

(The choice of the notation for the constant will turn out to be convenient). Given a real-valued k, the solution for $a(x)$ is a sinusoidal function, $a(x) = a_k \sin(kx + \theta_k)$ for arbitrary constant amplitude a_k and phase θ_k.

[1]The Airy theory as presented here should not be confused with the *Airy function* (although there are some relations) that describes the phenomenon that waves entering still water develop a 'first' wave with increased amplitude; this phenomenon is clearly noticed when generating a harmonic wave train in a wave tank, or in HAWASSI software.

Then $f(z)$ is a superposition of exponential functions $e^{\pm kz}$. The combination that satisfies the impermeability condition at the bottom $z=-D$, i.e. $f'(-D)=0$, is given by a hyperbolic cosine function, and the complete solution becomes

$$\Phi_k(x,z) = a_k \sin(kx-\theta_k)\frac{\cosh k(z+D)}{\cosh kD} \tag{2.7}$$

(The normalization in f such that $f(z=0)=1$ turns out to be handy in the following; then a_k is truly the amplitude of the potential at $z=0$: namely $\Phi_k(x,z=0) = a_k \sin(kx-\theta_k)$.)

This is a solution for any choice of (real-valued) k. Hence we can take a superposition, which will then also be a solution:

$$\Phi(x,z) = \int_0^\infty a_k \sin(kx-\theta_k)\frac{\cosh k(z+D)}{\cosh kD}dk$$

This can be rewritten in a somewhat more convenient way using complex notation as

$$\Phi(x,z) = \int_{-\infty}^\infty \alpha(k)\exp(ikx)\frac{\cosh k(z+D)}{\cosh kD}dk$$

with α the complex amplitude

$$\alpha(k) = |\alpha(k)|\exp(i\theta(k))$$

The phase information θ_k is now included as the argument of α. Note that to assure that $\Phi(x,z)$ is a real-valued function, the complex amplitude has to satisfy $\alpha(-k) = \overline{\alpha(k)}$, where the bar denotes complex conjugation.

The following observation is of crucial importance: for $z=0$ we get

$$\Phi(x,z=0) = \int_{-\infty}^\infty \alpha(k)\exp(ikx)\,dk$$

and so an important result is obtained

COROLLARY 2. *The solution of the Laplace problem $\Delta\Phi=0$ in a strip $z\in(-D,0)$ with vanishing normal derivative at the bottom and at lateral boundaries is uniquely determined by the* Dirichlet boundary value $\Phi(x,z=0)=\phi(x)$ *at $z=0$ and given by*

$$\Phi(x,z) = \int_{-\infty}^\infty \hat\phi(k)\exp(ikx)\frac{\cosh k(z+D)}{\cosh kD}dk \tag{2.8}$$

where $\hat\phi(k)$ is the Fourier transform of $\phi(x)$. Moreover, the normal derivative at the upper boundary, the so-called Neumann boundary value, is given by

$$\partial_z\Phi(x,0) = \int_{-\infty}^\infty \hat\phi(k)\,k\tan(kD)\exp(ikx)\,dk \tag{2.9}$$

which defines the so-called Neumann-to-Dirichlet *operator.*

The importance is that *also for more general cases, i.e. non-flat upper surface and varying bottom, there is (under reasonable smoothness conditions) a unique solution of the Laplace problem for given value of the potential at the deformed upper boundary,* despite the fact that the solution cannot be written down explicitly. Clearly this result is important for wave theory since then the upper surface is not flat.

3. Continuity = mass conservation equation

Without the restriction to potential flows, we will now consider the equation for the free surface of a layer of *incompressible* fluid above an impermeable bottom at $z = -D(x)$.

We will assume that the surface can be described as the graph of a function. Restricting to 2D for notational simplicity, this means that we can describe the surface as a function

$$z = \eta(x,t)$$

We will present two different formulations, and show that they are equivalent for incompressible flows. In both cases the results are just a mathematical description of relation between quantities that are a consequence of the definition of these quantities. Hence, no physical laws are involved, and we talk about a so-called *kinematic relation.*

3.1. Continuity equation. This relation (which even holds also for compressible fluids!) is a sole consequence of the *definition* of the fluid surface. In the continuum description of the fluid by Lagrange, the dynamics is given by the position of the infinitely many particles, which have changing positions in time $(x(t), z(t))$; the velocity of the particles is related to the Eulerian description of the velocity at a fixed point in space as

$$\dot{x}(t) = U(x(t), z(t), t)$$
$$\dot{z}(t) = W(x(t), z(t), t)$$

Particles at the surface remain at the surface as a consequence to prevent discontinuities. Then, since the fluid surface is given by $z = \eta(x,t)$, following a particle at the surface, we get $z(t) = \eta(x(t), t)$, which after time differentiation leads to

$$\dot{z} = \partial_t \eta(x,t) + \partial_x \eta . \dot{x}$$

Using $\dot{x} = U, \dot{z} = W$ we get $\partial_t \eta(x,t) = W - \partial_x \eta . U$, which can be rewritten as

$$\partial_t \eta(x,t) = \mathbf{U} \cdot \mathbf{N} \text{ with } \mathbf{N} = (-\partial_x \eta, 1) \text{ at } z = \eta \tag{2.10}$$

Note that $\mathbf{n} = \mathbf{N}/|\mathbf{N}|$ is the outward pointing normal to the surface.

For potential flow we get

$$\boxed{\partial_t \eta(x.t) = \partial_N \Phi \text{ at } z = \eta} \tag{2.11}$$

with

$$\partial_N \Phi \equiv \nabla \Phi \cdot \mathbf{N}$$

Except for the normalization, $\partial_N \Phi$ is the *normal derivative of the potential* at the free surface.

3.2. Mass conservation. Another way to envisage the kinematic equation is to write it as a *balance law* as a consequence of the incompressibility of the fluid.

Consider a vertical slab in the water for $x \in (a,b)$ and from the bottom to the surface, and consider the change of the amount of water in the slab in a small time interval Δt which is given by the difference between the *flux* $\int_{-D}^{\eta} U dz$ through $x = b$ and through $x = a$

$$\left[\int_{-D(b)}^{\eta(b,t)} U(b,z,t)\, dz - \int_{-D(a)}^{\eta(a,t)} U(a,z,t)\, dz \right] \Delta t$$

Because of incompressibility, this change must be equal to (minus) the change of the surface, which is given by

$$-\int_a^b \left[\eta\left(x,t+\Delta t\right)-\eta\left(x,t\right)\right]dx$$

Equating both expressions, division by Δt and taking the limit $\Delta t \to 0$ leads to the *global mass conservation law*

$$\begin{aligned}\int_a^b \partial_t\eta\left(x,t\right)dx &= -\int_{-D(b)}^{\eta(b,t)} U\left(b,z,t\right)dz - \int_{-D(a)}^{\eta(a,t)} U\left(a,z,t\right)dz \\ &= -\,\partial_x \int_{-D(x)}^{\eta(x,t)} U\left(x,z,t\right)dz\Bigg]_{x=a}^{x=b}\end{aligned}$$

Realizing that the above holds for all a and b, taking the limit $b \to a$ the *local mass conservation law* is obtained

$$\partial_t\eta\left(x,t\right) = -\partial_x \int_{D(x)}^{\eta(x,t)} U\left(x,z,t\right)dz \tag{2.12}$$

Hence, the continuity equation has the interpretation to specify the local balance between surface height and variations of the total (depth integrated) momentum. In a more global interpretation, it is the *mass balance*: the change in water height is determined by the difference of influx and outflux through the vertical boundaries.

3.3. Proof of Equivalence. The above two expressions, with two completely different ways of derivation, are in fact the same for incompressible fluid, as shall now be shown. To that end observe

$$\partial_x \int_{-D(x)}^{\eta(x,t)} U\left(x,z,t\right)dz = \partial_x\eta.U\left(x,\eta,t\right) + \partial_x D.U\left(x,-D,t\right) + \int_{-D}^{\eta} \partial_x U\left(x,z,t\right)dz$$

Now using the incompressibility $\partial_x U = -\partial_z W$, and integrating over the vertical there results

$$\begin{aligned}\partial_x \int_{-D(x)}^{\eta(x,t)} U\left(x,z,t\right)dz &= \partial_x\eta.U\left(x,\eta,t\right) - W\left(x,\eta,t\right) + \partial_x D.U\left(x,-D,t\right) - W\left(x,-D,t\right) \\ &= -\mathbf{U}\left(x,\eta,t\right)\cdot\mathbf{N} + \mathbf{U}\left(x,-D,t\right)\cdot\mathbf{N}_{bot}\end{aligned}$$

with $\mathbf{N}_{bot}$ the (outward pointing) normal direction at the bottom. Since the bottom is impermeable $\mathbf{U}\left(x,-D,t\right)\cdot\mathbf{N}_{bot} = 0$, and we get

$$\partial_x \int_{-D(x)}^{\eta(x,t)} U\left(x,z,t\right)dz = -\mathbf{U}\left(x,\eta,t\right)\cdot\mathbf{N}$$

proving the equivalence of the two formulations for incompressible fluids.

3.4. Other formulations. It is tempting to introduce the vertical average of the horizontal velocity

$$\bar{U}\left(x,t\right) = \frac{1}{H}\int_{-D}^{\eta} U\left(x,z,t\right)dz$$

with H the so-called *total depth*

$$H\left(x,t\right) = D\left(x\right) + \eta\left(x,t\right)$$

Then the balance law becomes

$$\partial_t \eta = -\partial_x \left[H(x,t)\, \bar{U}(x,t) \right]$$

This is the formulation obtained in the shallow water equations to be derived further on when it is assumed that $\bar{U}$ does not depend on z. Then an easy coupling with the Bernoulli equation can be obtained, which is not immediate if in the general case U depends on z.

4. Momentum equation

A second dynamic equation at the free surface is obtained by looking at the physics that governs the dynamics of the internal fluid and that of the surface.

Starting point is Euler's equation in the interior, a generalization of Newtonian mechanics to continua

$$\frac{D}{Dt}\mathbf{U} = -g\mathbf{e}_3 \tag{2.13}$$

with the total time derivative given by

$$\frac{D}{Dt}\mathbf{U} \equiv \partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\,\mathbf{U}$$

Specified for potential flow we get after some manipulations

$$\nabla\left[\partial_t\Phi + \frac{1}{2}\left|\nabla\Phi\right|^2 + gz\right] = 0 \text{ in the fluid}$$

so that

$$\left[\partial_t\Phi + \frac{1}{2}\left|\nabla\Phi\right|^2 + gz\right] = \text{ constant in the fluid}$$

The quantity in brackets is known as (minus) the *Bernoulli pressure*

$$p_{Bernoulli} = -\left[\partial_t\Phi + \frac{1}{2}\left|\nabla\Phi\right|^2 + gz\right] \tag{2.14}$$

consisting of the sum of the so-called dynamic pressure, the nonlinear part of the pressure and the (hydro-) static pressure.

As a consequence, if a pressure free atmosphere above the water is assumed, $p_{Bernoulli} = 0$ at the surface, the 'constant' vanishes and we get the so-called *Bernoulli equation* in the interior

$$\partial_t\Phi + \frac{1}{2}\left|\nabla\Phi\right|^2 + gz = 0 \text{ in flow domain}$$

and just as well at the water surface

$$\boxed{\partial_t\Phi + \tfrac{1}{2}\left|\nabla\Phi\right|^2 + g\eta = 0 \text{ at } z = \eta} \tag{2.15}$$

5. Evolution equations

Summarizing, we get the complete formulation of the Euler equations for incompressible free surface flow as

$$\left\{\begin{array}{lll} \partial_t \eta = \mathbf{U}\cdot\mathbf{N} & \text{at} & z=\eta(x,y,t) \\ \partial_t \mathbf{U} = -(\mathbf{U}\cdot\nabla)\mathbf{U} - g\mathbf{e}_3 & \text{for} & -D(x,y) < z < \eta(x,y,t) \\ \nabla\cdot\mathbf{U} = 0 & \text{for} & -D(x,y) < z < \eta(x,y,t) \\ \mathbf{U}\cdot\mathbf{N} = 0 & \text{at} & \text{bottom and lateral boundaries} \end{array}\right. \tag{2.16}$$

Assuming irrotational flow, we get the summary for the *free surface potential flow* as

$$\boxed{\left\{\begin{array}{lll} \partial_t \eta = \partial_N \Phi & \text{at} & z=\eta(x,y,t) \\ \partial_t \Phi = -g\eta - \frac{1}{2}|\nabla\Phi|^2 & \text{at} & z=\eta(x,y,t) \\ \Delta\Phi = 0 & \text{for} & -D(x,y) < z < \eta(x,y,t) \\ \partial_n \Phi = 0 & \text{at} & \text{bottom and lateral boundaries} \end{array}\right.} \tag{2.17}$$

5.1. Time stepping for potential flows. The procedure to advance (numerically) the governing fully nonlinear equations is as follows.

To advance from time t_0 to time $t_1 = t_0 + \Delta t$, with Δt small, let η_0, Φ_0 be the elevation and potential at the surface at time t_0.

Then determine the right hand sides of the dynamic equations as follows

- From the given surface potential $\phi_0 = \Phi_0(x, \eta_0(x), t_0)$, calculate the potential $\Phi_{1/2}$ in the interior (with upper boundary determined by η_0 which is known); this solution is uniquely determined.
- Calculate $\partial_N \Phi_{1/2}$ for the kinematic equation and $\frac{1}{2}\left|\nabla\Phi_{1/2}\right|^2$ at the surface for the dynamic equation.
- Then the rhs's are determined, and the update at time t_1 can be obtained: the values of $\eta_1, \Phi_1|_{z=\eta}$ at the later time t_1 can be calculated.

5.2. Linearized equations and dispersion relation. By neglecting nonlinear effects, the surface equations become a linear system of equations that can be written with the surface potential $\phi(x,t) = \Phi(x, z=0, t)$

$$\left\{\begin{array}{ll} \partial_t \eta = \partial_z \Phi & \\ \partial_t \phi = -g\eta & \text{at } z=0 \end{array}\right. \tag{2.18}$$

Dispersion relation:

With these linearized dynamic equations, and Airy's expression for the potential, we are now able to derive the dispersion relation. Take for simplicity one harmonic wave, arbitrary real k. If $\phi(x,t) = a\sin(kx-\omega t)$, then from the momentum equation $\partial_t\phi = -g\eta$ we find $\eta(x,t) = \frac{\omega a}{g}\cos(kx-\omega t)$. Then $\partial_t\eta = \frac{\omega^2 a}{g}\sin(kx-\omega t)$. On the other hand, Airy's theory for the given ϕ gives $\partial_z\Phi|_{z=0} = k\tan k(kD)\, a$. Then the kinematic equation can only be satisfied (for $a \neq 0$) provided

$$\boxed{\omega^2 = gk\tanh(kD)} \tag{2.19}$$

which is the formula for the dispersion relation given before in Chapter 1.

5.3. Equations in surface variables. The time stepping algorithm above describes that the whole process is governed by two functions: η, the surface elevation, and ϕ the potential at the free surface. This can be made explicit. Indeed, by writing $\phi(x,t) = \Phi(x, z = \eta(x,t), t)$ it holds that

$$\partial_t \phi = \left[\partial_t \Phi + \partial_z \Phi \partial_t \eta\right]_{z=\eta}$$

and hence, the dynamic free surface equation can also be written (using the kinematic equation) as

$$\partial_t \phi = -g\eta - \frac{1}{2}\left|\nabla\Phi\right|^2 + \partial_z \Phi . \partial_N \Phi \text{ at } z = \eta(x,t)$$

So, we get the complete dynamic system

$$\left\{\begin{array}{lll} \partial_t \eta = \partial_N \Phi & \text{at} & z = \eta \\ \partial_t \phi = -g\eta - \frac{1}{2}\left|\nabla\Phi\right|^2 + \partial_z \Phi . \partial_N \Phi & \text{at} & z = \eta \end{array}\right. \tag{2.20}$$

This form of the equations may look not very attractive; we will recover this form as the Hamiltonian formulation in the next Chapter, interpreted in a more appealing way.

5.4. Shallow water equations. The shallow water equations describe an approximate model for waves that are long compared to the depth; the equations are nonlinear but do not have the property of dispersion: *all linear waves travel with the same velocity.*

For shallow water (long waves) it seems natural to assume that $\Phi(x,z,t) \approx \phi(x,t)$ because the shallow depth will lead to a flow that is mainly horizontal.

The equations are given as

$$\left\{\begin{array}{l} \partial_t \eta(x.t) = -\partial_x\left[(D+\eta)\,\partial_x \phi\right] \\ \partial_t \phi(x.t) = -g\eta - \frac{1}{2}\left(\partial_x \phi\right)^2 \end{array}\right. \tag{2.21}$$

These are the shallow water equations (SWE). Note that the continuity equation is in the desired form of a balance law. In this approximation, $\partial_x \phi = u$ is the approximate averaged *horizontal* velocity, while in the full equations it would be the tangential velocity.

Often the equations are described using u instead of the surface potential ϕ; then the equations read

$$\left\{\begin{array}{l} \partial_t \eta = -\partial_x\left[(D+\eta)\,u\right] \\ \partial_t u = -\partial_x\left[g\eta + \frac{1}{2}u^2\right] \end{array}\right. \tag{2.22}$$

We summarize a few special cases of these shallow water equations

5.4.1. *Linear model.* Upon neglecting all quadratic terms in the basic variables in the SWE above we get the linear system

$$\left\{\begin{array}{l} \partial_t \eta = -\partial_x(Du) \\ \partial_t u = -g\partial_x \eta \end{array}\right. \tag{2.23}$$

This set of 2 first order (in time) equations can now be rewritten as one second order equation in the surface elevation only:

$$\partial_t^2 \eta = \partial_x c^2 \partial_x \eta, \text{ with } c^2(x) = gD(x) \tag{2.24}$$

5.4.2. *Linear, Flat bottom.* For flat bottom $D = D_0$ we get that

$$\left\{ \begin{array}{l} \partial_t \eta = -D_0 \partial_x u \\ \partial_t u = -g \partial_x \eta \end{array} \right. \quad \text{or } \partial_t^2 \eta = c_0^2 \partial_x^2 \eta, \text{ with } c_0^2 = g D_0 \tag{2.25}$$

Note that for flat bottom, $c_0^2 = gD_0$ is indeed the square of the phase velocity for long waves, which shows the long wave character of the equations.

This equation has as *general solution* a superposition of an arbitrary wave running to the right and one to the left $\eta(x,t) = F(x - c_0 t) + G(x + c_0 t)$ as we have seen in Chapter 1.

5.4.3. *SWE-2HD (multi-directional waves).* In a similar way as for the 1HD case we get the basic set of equations

$$\left\{ \begin{array}{l} \partial_t \eta = -\nabla \cdot ((D + \eta)\, \mathbf{u}) \\ \partial_t \mathbf{u} = -\nabla \left(g\eta + \frac{1}{2} |\mathbf{u}|^2 \right) \end{array} \right. \tag{2.26}$$

As a limiting case the linear equations are obtained

$$\left\{ \begin{array}{l} \partial_t \eta = -\nabla \cdot (D\mathbf{u}) \\ \partial_t \mathbf{u} = -g \nabla \eta \end{array} \right. \tag{2.27}$$

which is equivalent to the second order equation

$$\partial_t^2 \eta = \nabla \cdot \left(c^2 \nabla \eta \right) \tag{2.28}$$

In particular, above flat bottom we have in the linear approximation

$$\left\{ \begin{array}{l} \partial_t \eta = -D_0 \nabla \cdot \mathbf{u} \\ \partial_t \mathbf{u} = -g \nabla \eta \end{array} \right. \quad \text{or } \partial_t^2 \eta = c_0^2 \nabla \cdot \nabla \eta \equiv c_0^2 \Delta \eta \tag{2.29}$$

6. Energy conservation

The total energy is the sum of the potential energy which depends only on the surface elevation η, and the kinetic energy which depends on the velocity in the fluid domain under the given surface.

The expression for the potential energy of the water can be derived by considering a 'particle' with mass ρdz at height z measured vertically from the still water level, and horizontal position x. This has potential energy given by $(\rho dz)\, gz$; then integration over z (considering all particles under the surface at the position x produces $\int_0^{\eta(x)} \rho g z dz = \frac{1}{2}\rho g \eta^2(x)$. Integrating over x leads to the *potential energy* given by

$$\boxed{P(\eta) = \rho \int \tfrac{1}{2} g \eta^2 dx} \tag{2.30}$$

The kinetic energy of the mass particle is $(\rho dz)\,|\mathbf{U}|^2$, and so we obtain the *kinetic energy* as

$$\boxed{K = \rho \int \int_{-D}^{\eta(x,t)} \tfrac{1}{2} |\mathbf{U}|^2 \, dz dx} \tag{2.31}$$

and the total energy as

$$\boxed{E = \rho \int \int_{-D}^{\eta(x,t)} \tfrac{1}{2} |\mathbf{U}|^2 \, dz dx + \rho \int \tfrac{1}{2} g \eta^2 dx} \tag{2.32}$$

In all of the following we will take $\rho = 1$ as normalization for the constant mass density.

6.1. Local balance law. We write the total energy as the integral over horizontal space of the energy density $\mathcal{E}$ or as the full domain integral over $\mathcal{E}_{dens}$

$$E = \int \mathcal{E} dx \text{ with } \mathcal{E} = \int_{-D}^{\eta(x,t)} \mathcal{E}_{dens} dz \text{ with } \mathcal{E}_{dens} = \frac{1}{2}|\mathbf{U}|^2 + gz$$

Since we assume that we are dealing with ideal fluids, in particular no viscosity, we expect that the total energy is constant in time, provided suitable boundary conditions apply (such as no influx from outside in the spatial domain of investigation). In other words we expect that the energy will have a local balance law in the form

$$\partial_t \mathcal{E} = -\partial_x \mathcal{F} \text{ with } \mathcal{F} \text{ the so-called } \textit{energy flux} \tag{2.33}$$

6.1.1. *Intermezzo: energy velocity.* In many cases it will turn out that we can write the energy flux as

$$\mathcal{F}(x) = V(x)\,\mathcal{E}(x)$$

where $V(x)$ is a certain velocity. This is justified from a dimensional point of view. But of course this is almost a tautology: we can define V as the quotient

$$V(x) = \frac{\mathcal{F}(x)}{\mathcal{E}(x)}$$

provided $\mathcal{E}$ is strictly positive. It is then colloquially said that the "the energy density propagates with the energy velocity". To simplify matters, sometimes approximations are sought to be able to identify $V(x)$ with a simpler, if possible with a space independent velocity, say V_{av} so that

$$\partial_t \mathcal{E} \approx -V_{av} \partial_x E$$

which is the translation equation: E propagates approximately with the speed V_{av}. However, great care should be taken with such simplifications. Following are a few examples.

EXAMPLE 1. *Consider the* translation equation *(the simplest linear unidirectional wave equation)*

$$\partial_t \eta = -c\partial_x \eta$$

and $\mathcal{E} = \eta^2$ *as the energy. Then we find easily and exact*

$$\partial_t \eta^2 = -c\partial_x \eta^2$$

which is obvious since η *and therefore any decent function of* η *translates with this velocity. In particular, for linear unidirectional dispersive waves above flat bottom, each harmonic mode and just as well its energy density is propagated with the phase velocity of that mode.*

EXAMPLE 2. *Consider* Linear shallow water equations *above flat bottom with energy density* $\mathcal{E} = \frac{1}{2}\left(Du^2 + g\eta^2\right)$ *and find*

$$\partial_t \mathcal{E} = -\partial_x \left(gDu\eta\right) = -\partial_x \left(c^2 u\eta\right)$$

Note that now

$$\frac{\mathcal{F}}{\mathcal{E}} = c\frac{cu\eta}{\frac{1}{2}\left(Du^2 + g\eta^2\right)} = c\frac{2\sqrt{D}u\sqrt{g}\eta}{Du^2 + g\eta^2}$$

For a single harmonic mode running to the right it holds that $g\eta = cu$ *and hence* $cu\eta = g\eta^2 = Du^2$ *and we get* $\partial_t \mathcal{E} = -\partial_x \left(c\mathcal{E}\right)$*, but for a superposition of modes, and for bidirectional waves, it gets more complicated.*

EXAMPLE 3. *Above varying depth, for the unidirectional equation*

$$\partial_t \eta = -\sqrt{c\left(x\right)}\partial_x \sqrt{c\left(x\right)}\eta$$

it holds

$$\partial_t \left(\eta^2\right) = -\partial_x \left(c\left(x\right)\eta^2\right)$$

and for the linear shallow water bidirectional equation we get $E = \frac{1}{2}\left(D\left(x\right)u^2 + g\eta^2\right)$ *and find*

$$\partial_t \mathcal{E} = -\partial_x \left(c^2\left(x\right)u\eta\right)$$

Nonlinear terms in the equation make the situation a bit more complicated as can be seen below.

EXAMPLE 4. *For the nonlinear shallow water equations*

$$\begin{cases} \partial_t \eta = -\partial_x \left[\left(D + \eta\right)u\right] \\ \partial_t u = -\partial_x \left[g\eta + \frac{1}{2}u^2\right] \end{cases}$$

the energy density and flux are

$$\partial_t \left[\frac{1}{2}\left(D + \eta\right)u^2 + \frac{1}{2}g\eta^2\right] = -\partial_x \left\{\left[\left(D + \eta\right)u\right]\left[\frac{1}{2}u^2 + g\eta\right]\right\}$$

Note that since u *here is the averaged horizontal velocity the flux can be written in agreement with the general flux for incompressible Eulerian flow (see below) as*

$$\mathcal{F}_{SW} = \left[\left(D + \eta\right)u\right]\left[\frac{1}{2}u^2 + g\eta\right] = \int_{-D}^{\eta} u\left(\frac{1}{2}u^2 + gz\right)dz$$

6.2. Local balance law for the Euler dynamics. For the Euler equations we can derive in a straightforward way after some tedious work the conservation law

$$\partial_t \left(\int_{-D}^{\eta(x,t)} \frac{1}{2} |\mathbf{U}|^2 \, dz + \frac{1}{2} g\eta^2 \right) = -\partial_x \left(\int_{-D}^{\eta(x,t)} U \left(\frac{1}{2} |\mathbf{U}|^2 + gz \right) dz \right) \tag{2.34}$$

which can be written more compact as

$$\partial_t \left(\int_{-D}^{\eta(x,t)} \mathcal{E}_{dens} dz \right) = -\partial_x \left(\int_{-D}^{\eta(x,t)} U \mathcal{E}_{dens} dz \right) \text{ with } \mathcal{E}_{dens} = \frac{1}{2} |\mathbf{U}|^2 + gz$$

To calculate the change in time, in the differentiation with respect to time, the integrand but also the surface elevation in the upper part of the fluid domain of the kinetic energy has to be taken into account.

6.3. Energy conservation for potential flows. For potential flow, the balance law for the Eulerian energy conservation applies just as well. Energy conservation can be derived independent of the above as

$$\frac{dE}{dt} = \int \int \nabla\Phi \cdot \nabla \partial_t \Phi dx dz + \int \left[\frac{1}{2} |\nabla\Phi|^2 \right]_{z=\eta} \partial_t \eta dx + \int g\eta \partial_t \eta dx$$

Now using the vector identity $\nabla\Phi \cdot \nabla\Psi = \text{div}(\Psi\nabla\Phi) - \Delta\Phi.\Psi$ and Gausz' Theorem (namely that for any smooth vector field $\mathbf{a}$ it holds that $\int\int \text{div}(\mathbf{a}(x,z))\, dxdz = \int \mathbf{a} \cdot N dx$) and taking into account the conditions at bottom and lateral boundaries, we get

$$\begin{aligned} \frac{dE}{dt} &= \left[\int \int -\Delta\Phi.\partial_t\Phi dx dz + \int \partial_N \Phi.\partial_t \Phi dx \right] + \int \left[g\eta + \frac{1}{2} |\nabla\Phi|^2 \right]_{z=\eta} \partial_t \eta dx \\ &= \int \partial_N \Phi.\partial_t \Phi|_{z=\eta} dx + \int \left[g\eta + \frac{1}{2} |\nabla\Phi|^2 \right]_{z=\eta} \partial_t \eta \end{aligned}$$

Inserting the dynamic equations $\partial_t\Phi = -g\eta - \frac{1}{2}|\nabla\Phi|^2$ and $\partial_t\eta = \partial_N\Phi$ leads to

$$\frac{dE}{dt} = \int \partial_N \Phi. \left[-g\eta - \frac{1}{2} |\nabla\Phi|^2 \right]_{z=\eta} dx + \int \left[g\eta + \frac{1}{2} |\nabla\Phi|^2 \right]_{z=\eta} \partial_N \Phi$$

expressing the conservation in time of total energy

$$\boxed{\frac{dE}{dt} = 0} \tag{2.35}$$

7. Exercises

EXERCISE 1. *Check the form of the continuity equation as a* local balance law *from first principles by considering the conservation of water-mass flowing though a narrow tube from bottom to surface over an interval* $[x_1, x_2]$ *with* $x_2 = x_1 + \Delta x$*, over a small time* Δt*.*

EXERCISE 2. *The mapping* $\phi \to \partial_N \Phi$ *is called the* Dirichlet-to-Neumann operator*or (why?). Give the expression for the Dirichlet-to-Neumann operator for the linear theory with flat surface (Airy theory) and for SWE.*

EXERCISE 3. *Take a Gaussian hump as initial condition (zero initial velocity). How do you think the linear evolution will look like for the SWE? How for full dispersion? (Try it with Hawassi-software, linear and nonlinear with various choices for the dispersion.)*

EXERCISE 4. *A special (relatively simple but famous) example of a unidirectional, nonlinear wave model is the* KdV (Korteweg-de Vries) equation*, in normalized form:*

$$\partial_t \eta = -c_0 \partial_x \left(\eta + \alpha \partial_x^2 \eta + \beta \eta^2 \right)$$

where α, β *are certain constants and* $c_0 = \sqrt{gD}$*. What is the dispersion relation for the linearized model? For what choice of* α *do we get the correct third order term of the exact dispersion relation?*
Write this equation in (generalized) Hamiltonian form by finding the Hamiltonian $H_{KdV}(\eta)$ *so that*

$$\partial_t \eta = -c_0 \partial_x \delta_\eta H(\eta)$$

Show that the energy is conserved (i.e. $H(\eta)$ *is constant).*

CHAPTER 3

Hamiltonian-Boussinesq wave formulation

The previous chapter described the basic pde-formulation of the water wave problem, a quite common way to look at the dynamics.

In this chapter the different approach on which the HAWASSI software is based is explained. We will not go into technical details but emphasize the main ideas and advantages.

The fact that Φ is completely determined by specifying its value at the free surface $z = \eta$, as stated in the previous chapter, section 2 and shown for the geometry of a strip, leads to the question if the full dynamics can simply be described in ϕ, η variables. This is determined by the fact if it is possible to calculate, or approximate, the values of the quantities at the right hand sides of the dynamic equations directly from ϕ, η. If possible, this would lead to a dimension reduction since then it is not necessary to calculate the internal fluid flow.

Many variants, generally known as *Boussinesq approximations*, have been derived using perturbation methods (with assumptions on wave height and wave lengths) to express the rhs's in ϕ, η. An essential problem that is difficult to overcome is the dispersion relation; the non-algebraic expression (2.19) cannot be easily transformed to (discretized) differential operators.

HAWASSI software is based on the property that it is possible to find a good approximation of the kinetic energy $K(\phi, \eta)$ for potential flows. The background of this is that the equations (2.20) are in fact a *Hamiltonian system*, a generalization to infinite dimensions of the structure of many conservative systems from Classical Mechanics. This will be described in the next sections, after an illustration of the Hamiltonian structure in finite dimensions.

1. Finite dimensional Hamiltonian systems

Suppose a dynamical system is described with a state variable u, a vector with p components: $\mathbf{u} \in \mathbb{R}^p$.

> The meaning of the word *state variable* in a dynamic system is that the variable contains all the information at a certain time, so that the future is uniquely determined.

For a first order differential equation (evolution equation)

$$\partial_t \mathbf{u} = F(\mathbf{u})$$

under mild conditions for F, the future evolution is indeed determined once an *initial value* for $\mathbf{u}$, say at time $t = 0$ a value $\mathbf{u}_0$, is specified. Now consider cases with special functions F. Namely F of the form $F(\mathbf{u}) = \Gamma \nabla E(\mathbf{u})$

$$\partial_t \mathbf{u} = \Gamma \nabla E(\mathbf{u}) \tag{3.1}$$

where Γ is a matrix and E a given function of $\mathbf{u}$, with $\nabla E(\mathbf{u})$ the gradient. Then investigate the change of the function E in time:

$$\frac{dE}{dt} = \nabla E \cdot \partial_t \mathbf{u} = \nabla E \cdot \Gamma \nabla E$$

Two special cases (formally speaking not completely disjoint classes) are as follows.

Steepest descent flow:

For the case that Γ is a symmetric matrix, that is moreover negative definite, the value of$E$ will decrease; a simple example is given for Γ minus the identity matrix, since then

$$\frac{dE}{dt} = \nabla E \cdot \Gamma \nabla E = -|\nabla E|^2$$

The quantity E is decreasing monotonically, in directions perpendicular to the level lines of E, i.e. in the directions of steepest descent. This dynamic behaviour is characteristic for *dissipative* systems.

Hamiltonian flow:

Now consider the case that Γ is any skew-symmetric matrix ($\Gamma^* = -\Gamma$); then necessarily $p = 2n$ is even. Since $\mathbf{a} \cdot \Gamma \mathbf{a} = 0$ for any vector $\mathbf{a}$, it follows that

$$\frac{dE}{dt} = 0 \text{ if } \Gamma^* = -\Gamma \tag{3.2}$$

Hence the quantity E is conserved: the dynamic trajectories remain at the level set of E determined by the initial condition: $E(\mathbf{u}(t)) = E(\mathbf{u}_0)$ for all time. Such systems are examples of *conservative systems*.

A special case of conservative systems is obtained for systems from *Classical Mechanics*. Then the state vector consists of a position vector $q \in \mathbb{R}^n$ and a momentum vector $p \in \mathbb{R}^n$. The total energy of the system denoted by $H(q, p)$.

Dynamic systems of the form

$$\partial_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \nabla_q H \\ \nabla_p H \end{pmatrix} = \begin{pmatrix} \nabla_p H \\ -\nabla_q H \end{pmatrix} \tag{3.3}$$

with I_n the identity matrix in $\mathbb{R}^n$, are called (classical) Hamiltonian systems, with (q, p) the set of *canonical variables* and $H(q, p)$ the Hamiltonian. The characteristic form is common for all dynamic equations in

Classical Mechanics, and just as well for the surface wave equations as we shall see later.

EXAMPLE 5. *The simplest example is a mass particle with mass m that can move on a line, position denoted by q, under the influence of a force with potential $V(q)$. Then the momentum is $p = m\dot{q}$ and according to Newton $\dot{p} = -\partial_q V$. This can be rewritten as (3.3) with the Hamiltonian*

$$H(q,p) = \frac{1}{2m}p^2 + V(q)$$

which is the total energy, the sum of kinetic and potential energy.

EXAMPLE 6. *Pendulum equation*
Consider a mass point of mass m attached to a massless non-extendable cord of length ℓ under the influence of gravity g. The angle of the cord with the downward vertical is denoted by θ.
The equation for the pendulum motions are found from Newton's equation

$$m\ell^2\ddot{\theta} = -mg\ell\sin\theta,$$

i.e. for $\omega^2 = g/\ell$ by

$$\ddot{\theta} = -\omega^2\sin\theta.$$

For this last equation the total energy E is the sum of potential P and kinetic energy K:

$$E = K + P, \text{ with } K = \frac{1}{2}\dot{\theta}^2 \text{ and } P = \omega^2(1-\cos\theta).$$

This energy is conserved for all possible motions:

$$\frac{d}{dt}E = 0 \text{ for solutions.}$$

Introducing the velocity type of variable $v = \dot{\theta}$, the equations can be written as a system of two first order equations for the state variable (θ, v)

$$\begin{cases} \dot{\theta} = v, \\ \dot{v} = -\omega^2\sin\theta, \end{cases}$$

Writing the total energy as function of θ, ν

$$E = H_{pen}(\theta, v) = \frac{1}{2}v^2 + \omega^2(1-\cos\theta).$$

the equations are of the form of a Hamiltonian system:

$$\begin{cases} \dot{\theta} = \partial_v H_{pen}(\theta, v), \\ \dot{v} = -\partial_\theta H_{pen}(\theta, v). \end{cases} \tag{3.4}$$

This shows that using the correct variables, the expression for the energy alone provides all the information for the dynamical equations when these are written in the characteristic form (3.4).
The form of the Hamilton equations immediately imply energy conservation, whatever the precise form of the Hamiltonian, as follows from

$$\begin{aligned} \frac{d}{dt}H_{pen}(\theta, v) &= \partial_\theta H_{pen}.\dot{\theta} + \partial_v H_{pen}.\dot{v} \\ &= \partial_\theta H_{pen}.\partial_v H_{pen} + \partial_v H_{pen}.(-\partial_\theta H_{pen}) = 0. \end{aligned}$$

EXAMPLE 7. *Harmonic oscillator.*
The pendulum equation above is often studied for small deviations (small angles) only. Then the usual argument is to replace the driving force by its linear approximation: using $sin\theta \approx \theta$ *for small* θ*, the equation becomes the so-called harmonic oscillator*

$$\ddot{\theta} = -\omega^2\theta$$

this name because the solutions of this equation are the harmonic functions $sin(\omega t)$ *and* $cos(\omega t)$*.*
This equation, the linearization of the pendulum equation, is again of Hamiltonian form, now with Hamiltonian:

$$H_{lin}(\theta, v) = \frac{1}{2}v^2 + \frac{1}{2}\omega^2\theta^2.$$

Hence for the harmonic oscillator motions, this H_{lin} *is conserved.*

Looking at the last 2 examples, we can also reverse the argument:

- Start with the original pendulum Hamiltonian, $H_{pen} = \frac{1}{2}v^2 + \omega^2(1 - \cos\theta)$
- Approximate H_{pen} by quadratic terms alone. Then using the fact that $\cos(\theta) \approx 1 - \frac{1}{2}\theta^2$ we get precisely H_{lin}:

$$H_{pen} \approx H_{lin}$$

- Obtain the linearized pendulum model, i.e. the harmonic oscillator, as the Hamiltonian system (3.4) with H_{lin}.

If we interpret the nonlinear pendulum equation as the 'difficult' problem, and the linearized harmonic oscillator equation as the simplified, easier, model, we can formulate the above methodology as a Hamiltonian consistent modelling method:

> **Hamiltonian consistent modelling method***:*
> *Given a (difficult) Hamiltonian system, we can obtain a simplified model, with guaranteed Hamiltonian structure (and approximate Hamiltonian (energy) conservation as consequence), by simplifying the Hamiltonian.*

2. Dimension reduction and Hamiltonian formulation

In this section it will be shown that the water wave problem (an infinite dimensional problem) has the same dynamic Hamiltonian structure as the pendulum example in the previous section. Now the surface elevation η will play the role of position, and the potential at the free surface ϕ the role of momentum; the gradients $\nabla_{q,p}H$ are then being replaced by variational derivatives of the total energy function*al*.

To start with the variational calculus to introduce the variational derivatives, we first consider the potential energy. The kinetic energy will be considered after investigating Dirichlet's principle, which will be the basis of the Hamiltonian formulation, and the basis tot construct consistent approximations in HAWASSI software.

2.1. Potential energy. The potential energy of the water as derived before is given by

$$P(\eta) = \int \frac{1}{2} g\eta^2 dx \tag{3.5}$$

We now extend the well-known concept of differentiation of functions of one or more variables to this functional. (See the Annex for a more elaborate discussion of variational calculus.)

To that end, consider the change δP of the potential energy when the surface $\eta(x)$ changes with a (small) amount given by a *variation* $\delta\eta(x)$. The variation means a small change $\eta(x) \to \eta(x) + \delta\eta(x)$, using notation and terminology from the 18^{th} century. Then we get

$$\begin{aligned}\delta P = P(\eta+\delta\eta) - P(\eta) &= \int \frac{1}{2} g\left[(\eta+\delta\eta)^2 - \eta^2\right] dx \\ &= \int \frac{1}{2} g\left[2\eta\delta\eta + \delta\eta^2\right] dx\end{aligned}$$

Neglecting the second order term with $(\delta\eta)^2$, we get the so-called *first variation in the 'direction* $\delta\eta$ as

$$\delta P(\eta;\delta\eta) = \int g\eta.\delta\eta dx = \langle g\eta, \delta\eta\rangle$$

where $\langle,\rangle$ is the usual L_2-innerproduct. Comparing how the gradient of a multi-variable function $f:\mathbb{R}^n \to \mathbb{R}$ is defined using the innerproduct in $\mathbb{R}^n$, via a variation δx

$$f(x+\delta x) - f(x) \approx \langle \nabla f(x), \delta x\rangle_{\mathbb{R}^n}$$

we define the *variational derivative* with respect to η as a generalization of the gradient but now using the L_2-innerproduct. Writing $\delta_\eta P$ for the variational derivative, we get

$$\delta_\eta P(\eta) = g\eta.$$

2.2. Dirichlet's principle. The following basic fact has lead to the foundation of functional analysis and will be used to find consistent approximations of the kinetic energy further on:

> *Dirichlet's principle* states that among all possible interior flows Φ that have prescribed surface value ϕ, the (only) one that minimizes the so-called *Dirichlet functional* is actually the potential flow Φ that satisfies the Laplace equation in the interior and the no-flux boundary conditions on B.

In more detail, consider for a layer of water with given surface $z = \eta(x)$ and bottom $z = -D(x)$ the minimization problem for the Dirichlet functional $D(\Phi)$

$$\boxed{\min_{\Phi} \{\mathcal{D}(\Phi) | \, \Phi = \phi \text{ at } z = \eta\}} \tag{3.6}$$

$$\text{with } \mathcal{D}(\Phi) = \int \int_{-D}^{\eta(x)} \frac{1}{2} |\nabla \Phi|^2 \, dxdz \tag{3.7}$$

For a minimizer Φ it should hold that

$$\delta \mathcal{D}(\Phi, \delta\Phi) = 0 \text{ for all admissible variations } \delta\Phi$$

Taking variations $\delta\Phi$ we get, using Gausz' theorem

$$\begin{aligned} \delta \mathcal{D}(\Phi, \delta\Phi) &= \int \int \nabla\Phi . \delta \nabla \Phi dxdz \\ &= \left[\int \int -\Delta\Phi . \delta\Phi dxdz + \oint \partial_n \Phi . \delta\Phi dS \right] \end{aligned}$$

First consider variations $\delta\Phi$ that vanish at the boundary. Then the boundary integral vanishes. From the fact that interior variations $\delta\Phi$ are not restricted, we get from $\int\int -\Delta\Phi.\delta\Phi dxdz = 0$ for arbitrary variations $\delta\Phi$ that the minimizer satisfies

$$\Delta\Phi = 0 \text{ in the interior}$$

stating that the minimizing Φ is a potential flow, satisfying the Laplace equation.

Using this result we are left with

$$\oint \partial_n \Phi . \delta\Phi dS = 0 \text{ for all admissible variations } \delta\Phi \text{ at the boundary}$$

Since $\Phi = \phi$ is prescribed at $z = \eta$, the variations are restricted to satisfy $\delta\Phi = 0$ at $z = \eta$. Then the contour integral reduces to the integral over the bottom (and lateral boundaries). Since variations $\delta\Phi$ are not restricted at these boundaries, $\delta\Phi$ is free, and we get

$$\partial_n \Phi = 0 \text{ at } z = -D(x) \text{ and at lateral boundaries}$$

This is called a *natural boundary condition*: at these boundaries Φ was not prescribed (different from the free surface), and as a consequence of the minimization requirement, we get these such additional conditions for the minimizing potential. Note that the bottom boundary implies that the bottom is *impermeable* (as it should); the vanishing normal derivative at the lateral boundaries implies 'no-flux' conditions through these boundaries.

2.3. Kinetic energy. In order to derive the Hamiltonian formulation, let us first define the kinetic energy in its dependence on the surface variables because we want restrict to potential flows

$$\boxed{K(\phi, \eta) = \left\{ \int \int_{-D}^{\eta(x,t)} \tfrac{1}{2} |\nabla\Phi|^2 \, dxdz \;\middle|\; \begin{array}{llll} \Delta\Phi = 0 & \text{for} & -D < z < \eta \\ \Phi = \phi & \text{at} & z = \eta \\ \partial_n \Phi = 0 & \text{at} & B \end{array} \right\}} \tag{3.8}$$

In a similar way as done for the potential energy and Dirichlet's principle, we now consider the change of the kinetic energy for variations in both η and in ϕ

Since any variation $\delta\phi$ in the surface potential ϕ implies a unique corresponding variation $\delta\Phi$ of the potential on the whole fluid area, we get for the total variation of K from variations $\delta\phi, \delta\eta$ the result

$$\begin{aligned}
\delta K(\phi,\eta) &= \int\int \nabla\Phi.\delta\nabla\Phi dxdz + \int \left[\frac{1}{2}|\nabla\Phi|^2\right]_{z=\eta} \delta\eta dx \\
&= \left[\int\int -\Delta\Phi.\delta\Phi dxdz + \int \partial_N\Phi|_{z=\eta}.\delta\phi dx\right] + \int \left[\frac{1}{2}|\nabla\Phi|^2\right]_{z=\eta} \delta\eta dx \\
&= \int \partial_N\Phi|_{z=\eta}.\delta\phi dx + \int \left[\frac{1}{2}|\nabla\Phi|^2\right]_{z=\eta} \delta\eta dx \\
&\equiv \int \delta_\phi K.\delta\phi dx + \int \bar{\delta}_\eta K.\delta\eta dx
\end{aligned}$$

Note that from the first to the second line we used Gausz' theorem and the fact that $\partial_N\Phi = 0$ at the bottom and lateral boundaries, and from the second to the third line that $\Delta\Phi = 0$ in the interior.

The quantity $\delta_\phi K \equiv \partial_N\Phi|_{z=\eta}$ is the *partial variational derivative* with respect to ϕ; *partial*, because η is kept constant while giving the boundary value for Φ a variation $\delta\phi$.

The quantity $\bar{\delta}_\eta K$ is the change in K if for given $\Phi(x,z)$ the boundary gets a variation $\delta\eta$. However, this is not a partial derivative, because for fixed Φ the boundary value changes with changes in the boundary: $\Phi(x,\eta+\delta\eta) = \Phi(x,\eta) + \partial_z\Phi|_{z=\eta}.\delta\eta + ...$ The induced change of boundary value $\delta\phi = \Phi_z|_{z=\eta}.\delta\eta$ in the potential causes an additional change in K. This contribution has to be compensated, and we get for the partial derivative with respect to η

$$\delta_\eta K = \bar{\delta}_\eta K - \delta_\phi K.\Phi_z|_{z=\eta} = \left[\frac{1}{2}|\nabla\Phi|^2\right]_{z=\eta} - \delta_\phi K.\Phi_z|_{z=\eta} \tag{3.9}$$

Hence the partial *variational derivatives* $\delta_\phi K$ and $\delta_\eta K$ are defined and found to be

$$\delta_\phi K = \partial_N\Phi \tag{3.10}$$

$$\delta_\eta K = \left[\frac{1}{2}|\nabla\Phi|^2 - \Phi_z|_{z=\eta}\partial_N\Phi\right]_{z=\eta} \tag{3.11}$$

Now consider the *Hamiltonian* as the total energy expressed in ϕ, η as

$$\boxed{H(\phi,\eta) = K(\phi,\eta) + P(\eta)} \tag{3.12}$$

Then it can be observed that the surface equations from Chapter 2, (2.20), can be written as a Hamiltonian system

$$\boxed{\begin{cases} \partial_t\eta = \delta_\phi H & = \delta_\phi K \\ \partial_t\phi = -\delta_\eta H & = -[g\eta + \delta_\eta K] \end{cases}} \tag{3.13}$$

This shows that:

> The dynamical system formulation of the water wave problem in this form has the structure of a *Hamiltonian system*, with η, ϕ the *canonical position and momentum variable* respectively.

This specific formulation for potential flows was given by Zakharov 1968 and Broer 1974, without using the action principle (see next section) from which the Hamilton equations are obtained.

3. Pressure (action) principle

We have shown above that the Hamiltonian formulation of free surface waves is equivalent to the basic equations of Chapter 2, i.e. to the continuity equation and the momentum equation at the free surface.

There is another way to derive these equations from a *canonical action principle*, an idea that dates back to Maupertuis and Euler in the 18th century (see Annex on Variational Calculus). For water waves, the action principle is actually equivalent to a pressure principle, as follows.

Consider the dynamic *principle of stationary pressure*, i.e. the requirement that for arbitrary variations of $\Phi(x,z,t), \eta(x,t)$ the first variation vanishes of the *dynamic pressure functional*

$$\int_{t_1}^{t_2} dt \int \int_{-D}^{\eta} \left[-\partial_t \Phi - \frac{1}{2} |\nabla \Phi|^2 - gz \right] dz dx \tag{3.14}$$

Bateman 1928 used the fact that internal variations of Φ leads to potentials; Luke 1967 considered this functional and showed that variations of η lead in addition to the correct surface momentum equation.

By taking the time derivative in the integrand out of the volume integral (Miles 1976), up to an uninteresting total time derivative (giving contributions only at the end points of the time interval), there results

$$\int_{t_1}^{t_2} dt \int \left[\phi \partial_t \eta - \int_{-D}^{\eta} \left(\frac{1}{2} |\nabla \Phi|^2 + gz \right) dz \right] dx$$

By introducing the wave Hamiltonian in its dependence on the surface variables as above, this can be written as

$$\mathcal{A}_c(\eta, \phi) = \int_{t_1}^{t_2} dt \int \left[\phi \partial_t \eta - H(\phi, \eta) \right] dx \tag{3.15}$$

which is precisely of the form a *canonical action principle for surface waves.* Variations with respect to η, ϕ, now incorporating the time dependence, the equation for critical points are precisely the Hamilton equations for free surface waves derived from the basic equations in previous sections

$$\partial_t \eta = \delta_\phi H \text{ and } \partial_t \phi = -\delta_\eta H$$

4. Consistent Kinetic Energy approximations

Reflecting on the above, the original motivation to look for a dimension reduced formulation comes within reach. Essential is to have an explicit expression for $K(\phi,\eta)$. Although this cannot be achieved exactly, because there is no explicit expression for the potential Φ that corresponds to its surface value ϕ for domains different than a strip, any sufficiently good approximation of $K(\phi,\eta)$ will at least lead to a *consistent approximation* of the related equations. Consistent here means an approximation that shares the same Hamiltonian form and consequent important properties such as exact energy conservation.

4.1. Minimization property of the kinetic energy. In HAWASSI software, the approximations are obtained by exploiting the specific minimization property of the potential flows as expressed by Dirichlet's principle. From that result it is found that the kinetic energy (3.8) is just as well characterized by

$$K(\phi,\eta)=\min_{\Phi}\left\{\int\int_{-D}^{\eta(x)}\frac{1}{2}\left|\nabla\Phi\right|^{2}dxdz\,\middle|\,\Phi=\phi\text{ at }z=\eta\right\} \tag{3.16}$$

This minimizing property leads to the possibility to use Dirichlet's principle in a *constructive approximative way* by approximating the functional K (for which we cannot find a closed analytic description in ϕ,η) by a functional that is explicit in these variables, say $K_{approx}(\phi,\eta)$. Then we will use this approximate kinetic energy in the Hamiltonian dynamics.

In particular, in the continuum equation, the flux term $\partial_N\Phi$ in the right side is then approximated by

$$\delta_\phi K\equiv\partial_N\Phi\approx\delta_\phi K_{approx}$$

which defines the approximate Dirichlet-to-Neumann operator.

How a good approximation of $K(\phi,\eta)$ can be obtained is not immediately clear; direct numerical simulations will not be helpful. In HAWASSI software the approximations are obtained by restricting the set of minimizing functions in Dirichlet's principle, as will be illustrated in the next section.

REMARK 5. *Be aware that with Φ_{approx} the set of approximating functions, the expression $\partial_N\Phi_{approx}$ may* not *be a good approximation, while $\delta_\phi K_{approx}$ may be much better, as will be illustrated for the case of the shallow water equations later on.*

4.2. Generalized phase velocity operator. The observation that K is a quadratic expression in Φ leads to the conclusion that $K(\phi,\eta)$ must be a quadratic expression in ϕ. Even more so, since Φ and ϕ are not physical quantities themselves, because any constant can be added, it follows that $K(\phi,\eta)$ must be quadratic in $\partial_x\phi$ and hence has to be of the form

$$\boxed{K(\phi,\eta)=\tfrac{1}{2g}\int\left|C(\eta)\,\partial_x\phi\right|^{2}dx} \tag{3.17}$$

The factor g is taken so that the operator $C=C(\eta)$ can be interpreted as a *generalized phase velocity* as we shall see further on. All approximations will have this structure. As a special property, the variational derivative with respect to ϕ is given by

$$\delta_\phi K=-\partial_x\left(C^{*}C/g\right)\partial_x\phi$$

which guarantees that the continuity equation is in the form of a balance law, with $(C^*C/g)\,\partial_x\phi$ the horizontal momentum flux

$$\partial_t\eta = -\partial_x\left[(C^*C/g)\,\partial_x\phi\right] \tag{3.18}$$

5. Shallow water equations from consistent approximation

We illustrate the process of finding an explicit expression for an approximate $K(\phi,\eta)$ by restricting Dirichlet's principle for the limiting case of shallow water; in the next Chapter we illustrate the derivation of more advanced models that can be found in HAWASSI software.

> The shallow water equations is an approximate model for waves that are long compared to the depth; the equations are nonlinear but do *not* have the property of dispersion: all linear waves travel with the same velocity

For shallow water (long waves) it seems natural to assume that $\Phi(x,z) \approx \phi(x)$ because the shallow depth will lead to a flow that is mainly horizontal and then determined by the surface value ϕ of the potential. Taking this approximation, we can find the correct equations by using the derivation in the Hamiltonian consistent way.

Inserting the restriction $\Phi(x,z) \approx \phi(x)$ in Dirichlet's principle there results after integration with respect to z

$$K_{SW}(\phi,\eta) = \frac{1}{2}\int (D+\eta)\left(\partial_x\phi\right)^2 dx \tag{3.19}$$

Note that for this expression, the generalized phase velocity in (3.17) is now given by

$$C_{SW}(\eta) = \sqrt{g(D+\eta)} \tag{3.20}$$

The variational derivatives are found to be

$$\begin{cases} \delta_\phi K = -\partial_x\left[(D+\eta)\,\partial_x\phi\right] \\ \delta_\eta K = \frac{1}{2}\left(\partial_x\phi\right)^2 \end{cases} \tag{3.21}$$

so that the corresponding Hamilton equations are found as

$$\begin{cases} \partial_t\eta(x.t) = -\partial_x\left[(D+\eta)\,\partial_x\phi\right] \\ \partial_t\phi(x.t) = -g\eta - \frac{1}{2}\left(\partial_x\phi\right)^2 \end{cases} \tag{3.22}$$

These are the shallow water equations (SWE) considered before. Note that the continuity equation is in the desired form of a balance law.

Remark 6. *Observe the strength of the consistent approximation method: starting with an Ansatz in which the potential does* not *depend on z, namely $\Phi_{approx}(x,z) = \phi(x)$. Since then $\partial_N\Phi_{approx} = -\partial_x\phi.\partial_x\eta$, it can hardly be imagined that a good result can be achieved But this approximation leads to a good result after using it in the calculation of the approximate kinetic energy, which then leads correctly to the normal derivative from $\delta_\phi K$! (See the exercises.)*

Using instead of the surface potential the tangential fluid velocity $u = \partial_x\phi$, the equations

$$\begin{cases} \partial_t\eta = -\partial_x\left[(D+\eta)\,u\right] \\ \partial_t u = -\partial_x\left[g\eta + \frac{1}{2}u^2\right] \end{cases}$$

can also be written in a generalized Hamiltonian form as

$$\begin{cases} \partial_t\eta = -\partial_x\delta_u\bar{H}(u,\eta) \\ \partial_t u = -\partial_x\delta_\eta\bar{H}(u,\eta) \end{cases} \quad \text{with } \bar{H}(u,\eta) = \int \frac{1}{2}\left((D+\eta)\,u^2 + g\eta^2\right) dx$$

The energy conservation for the shallow water equations, that we proved earlier in Chapter2, is now immediate from the Hamiltonian character of the equations.

6. Exercises

EXERCISE 5. *From Bernoulli at the free surface we get the equation*

$$\partial_t \Phi = -gz - \frac{1}{2}\left|\nabla \Phi\right|^2 \text{ at } z = \eta\left(x,t\right)$$

How is that related to the Hamiltonian formulation $\partial_t \phi = \delta_\eta H\left(\phi,\eta\right)$ *of the dynamic surface equation for* $\phi\left(x,t\right) = \Phi\left(x,\eta\left(x,t\right),t\right)$*?*

EXERCISE 6. *Assume that a solution of the Dirichlet problem exists. Show that from Dirichlet's principle it follows that this is the only solution (uniqueness).*

EXERCISE 7. *Contemplate on the consistent modelling method of the kinetic energy via Dirichlet's principle as illustrated as follows. For the first variation of the kinetic energy with respect to* ϕ *we get*

$$\delta K\left(\Phi,\delta\Phi\right) = \int\int -\Delta\Phi.\delta\Phi dz dx + \int \partial_N \Phi.\delta\phi dx$$

Hence, if and only if $-\Delta\Phi = 0$ *for all variations of* $\delta\phi$*, it holds that* $\delta_\phi K = \partial_N \Phi$*. So for any approximative method, it will be the case that*

$$\delta_\phi K \neq \partial_N \Phi \text{ for approximations}$$

and the difference is determined by how well the Laplace equation is satisfied (i.e. how well the flow is modelled to be incompressible!).
In the shallow water approximation we take $\Phi\left(x,z\right) = \phi\left(x\right)$ *and there results after integration with respect to* z

$$\delta K\left(\Phi,\delta\Phi\right) = \int -\left(D+\eta\right)\partial_x^2\phi.\delta\phi dx + \int \partial_N\Phi.\delta\phi dx$$

so that with $\partial_N\Phi = \partial_z\Phi - \partial_x\Phi.\partial_x\eta = -\partial_x\phi.\partial_x\eta$ *we obtain for the variational derivative*

$$\delta_\phi K = -\left(D+\eta\right)\partial_x^2\phi - \partial_x\phi.\partial_x\eta \equiv -\partial_x\left(D+\eta\right)\partial_x\phi$$

which is the same result obtained by inserting the approximation $\Phi\left(x,z\right) = \phi\left(x\right)$ *directly in the Dirichlet integral and taking the variational derivative*

$$K\left(\Phi=\phi\right) = K_{SW}\left(\phi\right) = \frac{1}{2}\int \left(D+\eta\right)\left(\partial_x\phi\right)^2 dx \text{ with } \delta_\phi K_{SW}\left(\phi\right) = -\partial_x\left(D+\eta\right)\partial_x\phi$$

CHAPTER 4

HAWASSI wave models

The acronym HAWASSI stands for

Hamiltonian WAve - Ship - Structure Interaction.

There are two different models with their own specific implementation

HAWASSI – VBM Variational Boussinesq Model

HAWASSI – VBM1 for (1HD) simulations of irregular waves in wave tanks above flat and varying bottom and **VBM2** for (2HD) simulations with short-crested waves in coastal areas with harbours and strongly varying bathymetry, and for simulation of oceanic waves.

The interior fluid motion is modelled by a combination of a few (Airy-type) depth profiles; this makes it possible to optimize the dispersion properties depending on the specific case to be simulated. Nonlinear effects are accounted for in a weakly nonlinear way that is sufficient for most applications, and in a fully nonlinear way in a new version of the software.

Features of the 1&2HD code include

- The quality of dispersion is optimized for the specific wave problem to be simulated, which makes it possible to simulate deep ocean waves or very short waves (kh=15 or more) and infragravity waves
- Use of an unstructured grid (2HD) with mesh-size depending on bathymetry
- Wave influx from interior line using sources in the dynamic equations
- Use of efficient damping zones, (partially reflecting) walls for harbour lay-outs, etc.
- Calculation (1HD) of interior fluid velocities, fluid accelerations and the (dynamic and total) pressure at user defined positions between surface and bottom (as post-processing)

Facilities of the software include

- GUI for input of wave characteristics and model parameters, GUI for post processing
- Easy generation (using a GUI) of an unstructured 2HD mesh based on local depth for given bathymetry and geometric structures
- Input of wave properties (spectra, wave directionality) computed by (and in format of) SWAN to design irregular wave influx in the VBM2 model

- Time partitioned simulation to reduce hardware requirements
- Project examples with harmonic and irregular waves above bathymetry in various geometries
- Manual for easy operation and literature references to validations

HAWASSI –AB1 Analytic Boussinesq Model (1HD)

HAWASSI -AB for fast simulations of high, very nonlinear, breaking or non-breaking waves in 1HD (long crested), with the capacity to obtain reliable statistics about extreme wave occurrence in reasonable short time.

The interior fluid motion is modelled in an analytic way using Fourier-Integral Operators; as a consequence, dispersive properties are modelled exactly. Nonlinear effects can be included to any order, implemented until 4th-order. Breaking waves are modelled using an eddy-viscosity method with kinematic breaking criterion

Features of the 1HD code include

- Exact dispersion: waves of any wave length can be simulated
- Linear, 2nd, 3th and 4th order nonlinear; breaking
- Varying bottom, including run-up
- Various methods for wave influx from interior position
- Use of efficient damping zones and (frequency dependent, partially reflecting) walls
- Calculation of interior fluid velocities, fluid accelerations and the (dynamic and total) pressure at user defined positions between surface and bottom
- User-defined linear dispersion is accepted and then incorporated in the nonlinear terms

Facilities of the software include

- GUI for input of wave characteristics and model parameters, and GUI for post-processing
- Time partitioned simulation is possible to reduce hardware requirements
- Project examples with harmonic, focusing and irregular waves above bathymetry in various geometries
- Manual for easy operation and scientific description of the equations with literature references to the full theory and to more than 50 validations

1. Underlying Modelling Methods

As described in the previous chapter, the free surface dynamics for inviscid, incompressible fluid is governed by a set of Hamilton equations for the surface elevation and the surface potential. By approximating the kinetic energy functional (based on Dirichlet's principle) to obtain explicit dependence of the kinetic energy on the surface variables, the simulation of the interior flow is avoided, giving the much desired Boussinesq dimension reduction.

In VBM the interior flow is approximated by using a linear combination of a few vertical Airy profiles, characterized by the value of the chosen wave numbers. The choice of these values determines the dispersion relation, and optimized values are determined based on the spectrum of the influx signal (or initial profile) of the case under investigation. Therefore, VBM can have excellent, tailor-made, dispersive properties; deep water waves can be simulated just as well as infragravity waves.

In AB the kinetic energy is calculated using a generalization of the linear Airy potential; the generalization has exact dispersive properties for linear waves, can deal with arbitrary bottom and has flexible order of nonlinearity for the surface elevation.

For both codes, the Hamilton system conserves the (approximate) positive definite total energy exactly, avoiding sources of instability. The time dynamics is explicit, no CFL-conditions are required. Time stepping is done with matlab ode45 code, with automatic variable time step.

1.1. General aspects of HAWASSI implementations. For VBM, a Finite Element method using piecewise linear splines deals with the first order differentiations that appear in the approximate KE. In addition, a system of elliptic equations in the horizontal variables is solved for the amplitudes of the Airy functions. VBM uses variable grid size depending on the local depth, making 2HD simulations more efficient.

For AB, Fourier-integral operators (FIO) are used in a spatial-spectral implementation. The FIO's are approximated to enable efficient FFT (fast Fourier transform) -methods by interpolation techniques. Localization methods have been implemented to deal with walls, breaking waves, etc. AB, using Fourier methods, has uniform spatial grid step, and is very fast.

For both models a few items are treated in the same way and are described in this section.

1.2. Hamiltonian consistent spatial discretization. The HAWASSI models have the essential property that the dynamic equations are in Hamiltonian form and, as a consequence, have exact energy conservation, which could prevent instabilities and blow-up. This is an important part of the 'consistent' modelling approach.

In order to derive a numerical implementation, a time and space discretization has to be made of the dynamic equations. In HAWASSI that is done in such a way that the consistent approximation property is conserved:

(1) The time and space-discretization are completely decoupled;
(2) The explicit time integrator should be sufficiently accurate; in the HAWASSI models we choose to use RK45 with variable time step, as available in Matlab.

For continuous time, the spatially discretized system is again a Hamiltonian system, now in vector variables $\bar{\phi}, \bar{\eta}$, the discrete versions of ϕ, η, and with a discretized

Hamiltonian $\bar{H}\left(\bar{\phi},\bar{\eta}\right)$; the equations are in Hamiltonian form

$$\left(\begin{array}{c}\partial_t\bar{\eta}\\ \partial_t\bar{\phi}\end{array}\right)=\Gamma\left(\begin{array}{c}\partial_{\bar{\eta}}\bar{H}\\ \partial_{\bar{\phi}}\bar{H}\end{array}\right)$$

with Γ a skew symmetric matrix that will depend on the discretization method. The system is in Hamiltonian form, and as a consequence the discretized Hamiltonian $\bar{H}$ is conserved exactly. In more detail the description is as follows.

Instead of looking for functions $\eta\left(x,t\right),\phi\left(x,t\right)$ (from an infinite dimensional function space) we restrict to functions in a finite dimensional function space (dimension N), say

$$\eta\left(x,t\right)=\bar{\eta}\left(t\right)\cdot T\left(x\right)\equiv\Sigma_{j=1}^{N}\bar{\eta}_j\left(t\right)T_j\left(x\right) \tag{4.1}$$

where $T_j\left(x\right)$ form a set of independent base functions. Then the discretized Hamiltonian $\bar{H}$ is obtained by restriction

$$\bar{H}\left(\bar{\phi},\bar{\eta}\right)=H\left(\bar{\phi}\left(t\right)\cdot T\left(x\right),\bar{\eta}\left(t\right)\cdot T\left(x\right)\right) \tag{4.2}$$

The partial derivatives, for instance the gradient with respect to $\bar{\eta}$, denoted by $\partial_{\bar{\eta}}\bar{H}$, can then be found in a straightforward way from

$$\partial_{\bar{\eta}}\bar{H}\cdot\delta\bar{\eta}=\langle\delta H,\delta\bar{\eta}\cdot T\rangle=\langle\delta_\eta H,T\rangle\cdot\delta\bar{\eta}$$

leading to

$$\partial_{\bar{\eta}}\bar{H}=\langle\delta_\eta H,T\rangle \tag{4.3}$$

Taking the spatial innerproduct with T of the dynamic Hamilton equations, for instance

$$\langle\partial_t\eta,T\rangle=\langle\delta_\phi H,T\rangle$$

the restriction to the finite dimensional subspace leads to

$$\langle\partial_t\bar{\eta}\cdot T,T\rangle=\langle\delta_\phi H,T\rangle$$

which can be written as

$$M\partial_t\bar{\eta}=\partial_{\bar{\phi}}\bar{H}\text{ with }M_{jm}=\langle T_j,T_m\rangle$$

Similarly for the momentum equation.

M is the so-called *mass-matrix*, which is symmetric. Since the base functions are independent, M is also invertible and we get

$$\left(\begin{array}{cc}M&0\\0&M\end{array}\right)\left(\begin{array}{c}\partial_t\bar{\eta}\\ \partial_t\bar{\phi}\end{array}\right)=\left(\begin{array}{cc}0&1\\-1&0\end{array}\right)\left(\begin{array}{c}\partial_{\bar{\eta}}\bar{H}\\ \partial_{\bar{\phi}}\bar{H}\end{array}\right) \tag{4.4}$$

or

$$\left(\begin{array}{c}\partial_t\bar{\eta}\\ \partial_t\bar{\phi}\end{array}\right)=\bar{\Gamma}\left(\begin{array}{c}\partial_{\bar{\eta}}\bar{H}\\ \partial_{\bar{\phi}}\bar{H}\end{array}\right)\text{ with }\bar{\Gamma}=\left(\begin{array}{cc}0&M^{-1}\\-M^{-1}&0\end{array}\right) \tag{4.5}$$

Remark 7. *In Matlab time stepping procedures, the presence of a positive definite matrix acting on the time derivatives can be dealt with without the need to invert the matrix; so (4.4) can be dealt with directly.*

Example 8. *Finite element method with piecewise linear splines (used in VBM)*

Taking an equidistant grid in 1HD with mesh size Δx, piecewise linear elements are described by

$$T_k(x) = \max\left\{1 - \frac{|x - x_k|}{x_{k+1} - x_{k-1}}, 0\right\}.$$

Using these 'tent' functions, any function in x can be approximated by continuous piecewise linear functions. The canonical variables are then described with coefficients that depend on time

$$\eta(x,t) \sim \sum_{k=1}^{n} \eta_k(t) T_k(x), \quad \phi(x,t) \sim \sum_{k=1}^{n} \phi_k(t) T_k(x)$$

The mass matrix is explicitly given by its elements

$$M_{kj} = \int T_k(x)) T_m(x) dx$$

Since the highest derivatives in the Hamiltonian are of first order, only the first derivative of the base functions are needed, which are piecewise constant functions.

For instance, the simple Hamiltonian for linear SWE above bottom at depth $D(x)$, is then discretized as

$$\bar{H} = \frac{1}{2} G\bar{\phi} \cdot \bar{\phi} + g\bar{\eta} \cdot M\bar{\eta}$$

where G is the so-called stiffness matrix which is symmetric and has components

$$G_{km} = \int \partial_x T_k(x)).D(x)\, \partial_x T_m(x) dx$$

The consistently discretized Hamilton equations become

$$M\partial_t \bar{\eta} = \partial_{\bar{\phi}} \bar{H} = G\bar{\phi}$$
$$M\partial_t \bar{\phi} = -\partial_{\bar{\eta}} \bar{H} = -gM\bar{\eta}.$$

Explicitly, the matrices are given (above constant bottom D) by

$$M = \Delta x \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & & & & \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ & & & & \frac{1}{6} & \frac{1}{3} \end{pmatrix}, G = \frac{D}{\Delta x} \begin{pmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \cdots & \cdots & \cdots & & \\ & & \cdots & \cdots & \cdots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix}$$

Note that G can be recognized as the central finite difference approximation of the second derivative.

EXAMPLE 9. *Fourier truncation (used in AB)*

For an equidistant grid on an interval of length L with mesh size $\Delta x = L/(2N)$, we introduce wave numbers k_m and choose as base functions the harmonics

$$T_m(x) = e^{ik_m x}, \quad \text{with } k_m = m\frac{2\pi}{L}\Delta x = m\frac{2\pi}{N}$$

Continuous (periodic) functions on the interval are then approximated by a finite Fourier series

$$\eta(x) = \Sigma_{m=-N+1}^{N} \bar{\eta}_m e^{ik_m x}$$

In line with the complex base functions, the complex innerproduct is taken. Then $M = Id$ is the identity matrix. The linear SWE Hamiltonian above constant bottom becomes

$$\bar{H} = \frac{1}{2}\left[G\bar{\phi}.\bar{\phi} + g\left|\bar{\eta}\right|^2\right]$$

where now G =diag$\left(k_m^2\right)$.

1.3. Wave influxing and damping.

Wave influxing:

Wave influxing is done in an embedded way by adding a source to the rhs of the continuity equation. The general methodology is described in Lie ea. 2014.

A simple example is given by the unidirectional right travelling wave equation for which at $x = x_0$ a signal $s(t)$ is prescribed. Then adding a point source with time dependent strength $c_0 s(t)$ to the evolution equation as

$$\partial_t \eta = -c_0 \partial_x \eta + c_0 s(t)\, \delta_{Dirac}(x - x_0) \tag{4.6}$$

the resulting solution is a to the right travelling wave with the required signal at $x = x_0$

$$\eta(x,t) = s\left(t - \frac{x}{c_0}\right)\Theta(x - x_0)$$

where Θ is the Heaviside function

$$\Theta(\xi) = \begin{cases} 0 \text{ for } \xi < 0 \\ 1 \text{ for } \xi > 0 \end{cases}$$

It should be remarked that for more general cases the influx can be smeared out over an interval instead of from a single point, so-called *area influxing*. For any given spatial function $\gamma(x - x_0)$, for instance a narrow Gaussian function centered at the influx point, the strength of the influx signal has to be adjusted, from $s(t)$ to $\sigma(t)$ in such a way that the source $\gamma(x - x_0)\sigma(t)$ leads to the correct waves outside the generation area; it turns out that it is the group velocity that determines the total required strength in the generation area (illustrated by the presence of c_0 in the example above- which is accidentally also the phase velocity for SWE).

Damping zones:

Damping zones are used to prevent waves reaching the boundary of the numerical window if that is not desired. As is common, damping terms are added to the continuity and momentum equation. For instance for the unidirectional translation equation with damping term we have

$$\partial_t \eta = -c_0 \partial_x \eta - \alpha \eta \chi(x) \tag{4.7}$$

where α is a suitably chosen positive parameter, and $\chi(x)$ a function that vanishes outside the damping zone and smoothly increases to value 1 inside the damping zone.

Note the exponential decay for the simplified equation (actually in a moving frame) $\partial_t \eta = -\alpha\eta$ at a position inside the damping zone. In the damping zone the translating wave will then decay exponentially depending on the distance travelled, which depend therefore on c_0 (i.e. on the wave length in dispersive equations).

2. HAWASSI-VBM (Variational Boussinesq Model)

2.1. Consistent Hamiltonian derivation. We restrict to linear waves above flat bottom in the explanation of the basic ideas.

VBM Ansatz:

Motivated by Dirichlet's principle we restrict the potential (in the simplest version) to the sum of the surface potential $\phi(x)$ (as in shallow water case) and a term that has an a priori chosen depth-profile $F(z)$ with a space dependent coefficient $\psi(x)$

$$\Phi(x,z) = \phi(x) + F(z)\,\psi(x) \tag{4.8}$$

In order to assure that ϕ is the surface value and that Φ satisfies the bottom boundary condition above horizontal bottom, it is required to take

$$F(z=0) = 0, \quad \partial_z F(z=-D) = 0$$

We first investigate the consequences without specifying the function $F(z)$.

Approximate kinetic energy:

The kinetic energy is obtained by inserting the Ansatz in $\int\int |\nabla\Phi|^2\, dzdx$. Performing the integration over z the following result is obtained

$$\bar{K}(\phi,\psi) = \frac{1}{2}\int \left[D(\partial_x\phi)^2 + 2\beta\partial_x\psi\partial_x\phi + \alpha(\partial_x\psi)^2 + \gamma\psi^2\right] dx \tag{4.9}$$

Here α, β, γ are constants (in the linear theory) that depend on the choice of the function $F(z)$, explicitly given by

$$\alpha = \int_{-D}^{0} F^2 dz, \beta = \int_{-D}^{0} F(z)\, dz, \gamma = \int_{-D}^{0} F_z^2 dz \tag{4.10}$$

Not that in this linear limit, $\bar{K}$ does not depend on η.

From Dirichlet's principle we have to find the lowest possible value for $\bar{K}$. Since ψ is not specified yet, we choose it such that $\bar{K}$ is minimal with respect to ψ, for any given ϕ. This leads to a *compatibility condition*, i.e. a relation that couples ψ to the function ϕ

$$\delta_\psi \bar{K} \equiv -\partial_x\beta\partial_x\phi - \partial_x\alpha\partial_x\psi + \gamma\psi = 0 \tag{4.11}$$

This can be rewritten more clearly as

$$\mathcal{E}\psi \equiv -\alpha\psi_{xx} + \gamma\psi = \beta\phi_{xx}, \tag{4.12}$$

which is an elliptic equation for ψ, given ϕ.

Remark 8. *For the above Ansatz we get for the Laplacian*

$$\Delta\Phi = (\phi_{xx} + F\psi_{xx}) + F_{zz}\psi$$

Although desired, this expression cannot vanish identically for all x, z. Taking the weaker condition that

$$\int_{-D}^{0} F(z)\,\Delta\Phi dz = 0$$

should be satisfied is a kind of weak formulation *for the vanishing of $\Delta\Phi$ instead of the pointwise vanishing condition. Working out this condition leads precisely to the compatibility condition 4.12. The justification to use precisely this condition is given by the minimization argument from Dirichlet's principle.*

REMARK 9. *It can be shown that the elliptic operator is symmetric and positive definite, and is (hence) invertible. This implies that a unique ψ can be found depending on ϕ, written with the inverse of the elliptic operator $\mathcal{E}$ as $\psi = \mathcal{E}^{-1}(\beta\phi_{xx})$*

Hamiltonian system for VBM:

With the VBM Hamiltonian

$$H_{VBM}(\phi,\psi,\eta) = \bar{K}(\phi,\psi) + \frac{1}{2}\int g\eta^2 dx \tag{4.13}$$

we get for the kinematic and the momentum equation

$$\begin{cases} \partial_t\eta = \delta_\phi H_{VBM} & = -\partial_x\left[D\partial_x\phi + \beta\partial_x\psi\right] \\ \partial_t\phi = -\delta_\eta H_{VBM} & = -g\eta \\ \text{compatibility} & -\alpha\psi_{xx} + \gamma\psi = \beta\phi_{xx} \end{cases} \tag{4.14}$$

It is also possible to remove in a formal way the ψ-contribution from the formulation by explicitly writing $\psi = \mathcal{E}^{-1}(\beta\phi_{xx})$ and define the kinetic energy in η, ϕ as

$$K(\phi) = \bar{K}\left(\phi, \psi = \mathcal{E}^{-1}(\beta\phi_{xx})\right)$$

leading to

$$K(\phi) = \frac{1}{2}\int\left[D(\partial_x\phi)^2 - 2\beta\phi_{xx}\mathcal{E}^{-1}(\beta\phi_{xx}) + \gamma\left[\mathcal{E}^{-1}(\beta\phi_{xx})\right]^2\right]dx \tag{4.15}$$

Choices for profile function:

The choice of the profile function determines the dispersion relation (exercise).

In the first publications (Klopman ea. 2005) a parabolic profile function was taken, completely determined up to an arbitrary multiplicative factor by $F_{par}(z) = z(2D - z)$.

A better and more flexible choice is motivated by Airy's theory:

$$F_{Airy}(z) = \frac{\cosh\kappa(z+D)}{\cosh(\kappa D)} - 1$$

Note that then for $\psi = \phi$ we get a one-profile form of Airy's theory, now with space dependent amplitude

$$\Phi(x,z) = \phi(x)\frac{\cosh\kappa(z+D)}{\cosh(\kappa D)}$$

Here κ is an 'effective ' wave number, a parameter that can be chosen in an optimal way. Then dispersive properties can become much better with this vertical profile function: it is exact for the wave number $k = \kappa$ and very good for $k < \kappa$.

Extensions in HAWASSI VBM:

Using better Ansatzes for the potential Φ, the above can and has been extended, using essentially the fact that only the kinetic energy has to be improved, and the consequences are given by its variational derivatives in the free surface equations.

- More Airy vertical profiles, with different effective wave numbers $\kappa_{1,2}$

$$\Phi(x,z) = \phi(x) + F_1(z)\psi_1(x) + F_2(z)\psi_2(x) + ...$$

- Optimized choices of $\kappa_{1,2,...}$ in the vertical profile functions
- Nonlinear effects: function(s) $F = F(z,\eta)$ which leads to coefficients α, β, γ that will depend explicitly on η.

2.2. Numerical implementation. The most important aspects not yet mentioned of the numerical implementation for VBM are listed here.

- The compatibility equation, given by the elliptic eqn that couples ψ to ϕ, is positive definite and invertible. To solve this equation, various methods are available; in HAWASSI VBM a preconditioner is used.
- To optimize the effective wave number κ in the Airy profile, the method described in Lakhturov ea. is used; it minimizes the kinetic energy (in line with Dirichlet's principle) for the expected wave that corresponds to the given influx time signal.
- Also for extensions described above, the kinetic energy remains quadratic in the potentials, but the coefficients will depend on η in the nonlinear extension. This makes it possible to use the piecewise linear spline functions in 1HD and in 2HD on an unstructured grid also for the extensions.

3. HAWASSI-AB (Analytic Boussinesq)

3.1. Consistent Hamiltonian derivation. We first restrict to linear theory above flat bottom

AB Ansatz:

Motivated by Dirichlet's principle we restrict the potential to the explicit expression given by Airy's theory for a strip, the approximation for linear waves

$$\Phi(x,z) = \int \hat{\phi}(k) \frac{\cosh k(z+D)}{\cosh kD} e^{ikx} dk$$

As a consequence, all expressions will have to be formulated in Fourier space.

Approximate kinetic energy:

The kinetic energy $K(\phi)$ in the Hamiltonian can be obtained by direct calculation of $|\nabla\Phi|^2$ in Fourier space and integration over z; the result is

$$K_{ABlin}(\phi) = \frac{2\pi}{2} \int k \tanh(kD) \left|\hat{\phi}\right|^2 dk \tag{4.16}$$

(an additional factor 2π pops up because of Parseval's identity). Written as (3.17) we get the exact phase velocity of linear waves as the symbol of the phase velocity operator

$$K_{ABlin}(\phi) = \frac{1}{2g} \int |C_{AB}\partial_x \phi|^2 dx \text{ with } C_{AB} \hat{=} \sqrt{\frac{g\tanh(kD)}{k}} \tag{4.17}$$

The variational derivative $\delta_\phi K$ can be calculated to be

$$\delta_\phi K_{ABlin} = \int \hat{\phi}(k) k \tanh(kD) e^{ikx} dk$$

Hamiltonian system for AB model:

The resulting Hamiltonian system for the AB model with

$$H_{ABlin}(\phi,\eta) = K_{ABlin}(\phi) + \frac{1}{2}\int g\eta^2 dx \tag{4.18}$$

leads to the exact *linear* equations above flat bottom

$$\begin{cases} \partial_t \eta(x,t) = \delta_\phi H_{AB} & = \int \hat{\phi}(k) k \tanh(kD) e^{ikx} dk \\ \partial_t \phi = -\delta_\eta H_{AB} & = -g\eta \end{cases} \tag{4.19}$$

Extensions in HAWASSI AB:

The extension to include varying bottom and nonlinearity leads to the necessity to deal with *Fourier Integral Operators* (FIO's), a full treatment of which is out of the scope of this introductory text.

As a motivation we look at the Airy result for the potential, which is exact for linear waves above flat bottom. It is tempting to generalize this expression to include bottom variations and finite surface elevation by writing

$$\Phi(x,z,t) = \int \hat{\phi}(k) \frac{\cosh k(z+D(x))}{\cosh k(\eta(x,t)+D(x))} e^{ikx} dk \tag{4.20}$$

Note that now under the integral sign for the Fourier transform, the parameter x appears explicitly; hence a FFT is required for each x. This is an extension of pseudo-differential operators (PDO's) to Fourier Integral Operators (FIO's). Inserting this restriction into Dirichlet's principle, leads to complicated expressions. Approximations in any order of

nonlineairity are possible in principle; 2nd, 3th and 4th order approximations have been published and extensively tested (Kurnia & Van Groesen 2014, 2015). As an example, the 2nd order model can be effectively be written as (3.17)

$$K_{AB2}(\phi,\eta)=\frac{1}{2g}\int |C(\eta)\,\partial_x\phi|^2\,dx \text{ with } C(\eta)\hat{=}\sqrt{\frac{g\tanh kH(x,t)}{k}} \tag{4.21}$$

Here the *nonlinear phase velocity operator* is the FIO $C(\eta)$ and is a direct generalization of the phase velocity of the linear theory.

Accurate and rather fast simulations are possible with the various versions of nonlinear AB models; breaking has been included.

3.2. Numerical implementation. From its description in the previous section, it will be clear that for the AB-model the implementation will use Fourier truncation, which makes it possible to use the FFT techniques that are available.

The main points that have to be taken care of are the following

- Avoid '*aliasing*', which requires that all essential dynamics (taking into account the nonlinear mode generation) will take place in wave numbers below the Nyquest frequency
- Avoid '*periodic looping*': since Fourier truncation methods deal with functions on a finite **periodic** interval, waves travelling to the right (for instance) will reach the right end point and enter the left end point of the interval. This can only be prevented by damping zones that diminish the waves before reaching the boundaries of the interval.
- As stated above, for an operation with a FIO, a FFT is required at each spatial point in a direct implementation, which would make the implementation very time consuming. This has been avoided by approximating the FIO using *interpolation*. For instance, in (4.21) the expression is taken at a restricted number of x-values of $H(x,t)=\eta(x,t)+D(x)$; for each fixed value x in H the FIO becomes a simpler PDO; interpolating the PDO's with spatially dependent coefficients, can then reduce the numerical burden substantially.
- *Localization*. Fourier methods use global base functions, the harmonic functions, defined and non-vanishing over the entire interval, which means that each base function has an influence at all points. Conversely, a desired value at a specific point needs the subtle interaction of all base functions, very different from the local splines as in the VBM implementation. The best known example is that of a *delta-Dirac function*: to define it at one point, all Fourier modes should have an equally large contribution

  $$\delta_{Dirac}(x)=\frac{1}{2\pi}\int e^{ikx}dk$$

  This causes problems to introduce for instance a wall in the computational domain, or to introduce moving shore lines. These problems have been tackled and solved in the past years for these cases, and even for dynamic ships in waves, but it remains a topic that deserves much attention.

4. Exercises

EXERCISE 8. *Determine the dispersion relation for linear VBM using the parabolic profile and using one Airy profile. Discuss the quality of dispersion by comparing with the exact dispersion relation. Make plots of the dispersion relation compared to the exact dispersion.*

EXERCISE 9. *Show that the kinetic energy of VBM can be written as in (3.17) and verify that the operator is then indeed the phase velocity.*

EXERCISE 10. *Check with a direct calculation that the total energy is conserved.*

EXERCISE 11. *Show that the linear AB model has exact dispersion*

APPENDIX A

Tutorial: "The Art of Scientific Simulation"

Using a scientific simulation program as HAWASSI in a reliable way requires a lot of knowledge. The definition of many words has to be known, such as 'amplitude', 'peak period', 'irregular waves', 'spectrum', etc., all the concepts of Chapter 1.

More important is to realize that the underlying model (as any model will be) is only an approximation of reality. Knowing the consequences of the *assumptions and simplifications* that were made to design the model is essential (but often not enough) to be able to judge beforehand if a certain wave phenomenon can be captured or not. And even if a desired phenomenon (like non-linearity) can be captured, it may be only in a limited sense ('weakly', 2nd-order, higher order, etc.). On top of that, the calculations with the discretized versions of the continuous models will introduce numerical 'errors' that has to be understood and reduced as much as possible.

These introductory sentences are not meant to discourage the use of scientific software, on the contrary. Instead, it is a warning to be prepared and to remain *critical* about any input given, and about examining any output from the software. It can be seen as a challenge and fun to get the best results for the given software capabilities. Below are listed a few guidelines and general attitudes to help unexperienced users of software to approach quicker reliable results.

Embedded focusing:

In most cases you will aim to perform a 'difficult' simulation, such as irregular waves above varying bottom with nonlinear effects included. To find out what the capabilities of the software are, and to see the consequence of each separate difficulty, it is wise to *start with simplified cases*. This may seem to be a waste of time, but it is essential: at the end *it is the user who is responsible for judging the quality of the output.*

To achieve a good judgement, having results from various increasingly more complicated cases may make you more (or less) convinced about the quality of the final result. Stated differently, since Scientific Simulation will not provide 'exact' results, it is therefore insightful to envisage your end result as a 'point' that should be embedded, or being approached, by neighboring 'points' of other simulations, ensuring a kind of *model robustness*. Furthermore, it forces you to concentrate on the various effects separately, which may indicate quicker what the possible weak aspects in the final result could be. Besides that, changing numerical settings such as discretization size should give an equally important *numerical robustness*, see below, and adds to the points in the embedding of the final result.

Specified for the given 'difficult' example described above, it could start with the choice of the simulation domain, width of damping zones, boundaries, quality of influx procedure, etc. In many cases when the domain is not (completely) given, a most optimal choice (for fastest, yet accurate, simulations) and the quality of damping zones can already be found with linear simulations of harmonic waves (with period near the peak period of the desired irregular waves). The step to linear, irregular waves should

confirm the quality of damping zones and domain choices. Nonlinear effects can then be investigated by comparison with linear simulations.

Anticipated vs. actual result:

Strongly stated, a good advice would be: *do not perform simulations for cases for which the end result is not (qualitatively) known beforehand.* The user should know which effects will (or could) be important and should check that the software is described to be capable of covering these aspects, and to which degree; for instance whether steep waves that are expected to turn up can 'break' or not in the specific software. The 'Embedded focusing'– approach described above helps to identify the (strength of certain) effects, and the influence on the end result.

Thinking *in advance* what has to be expected also helps in studying the actual result, which is the output of the simulation: is it as expected or is it (much) different? In the last case one has to find out what is the cause, for instance wrong input, errors in the software (maybe a bug, or an effect that is not supported), or maybe the expectation was wrong (which should then lead to more insight in the phenomenon of wave propagation). In any case, having an anticipated result in mind helps to focus to look critical at the output, instead of being satisfied that 'some' result has been obtained.

Suggestions for chosing spatial grid-sizes:

A correct choice of the gridsize is a difficult matter since it depends strongly on the case to be simulated. Ideally, the gridsize should be fine enough to recover all physical phenomena in the simulation, and on the other hand as coarse as possible to reduce computational complexity and computation time.

Most important is that if you judge to have a good result for a certain grid size, a next step has to follow to verify if a finer gridsize produces qualitatively the same result that is quantitatively close to the original result; this aspect of *numerical robustness* is just as essential as the model robustness above.

To give some (necessarily rather vague) guidelines, we introduce the *minimal relevant wave length* (MRW) as the smallest wave length that is expected to be needed to cover the phenomena. Then the advice is to take

$$gridsize = \frac{MRW}{12}$$

so 'at least 12 points per relevant wave length'. The actual number '12 ' is not well founded; roughly speaking in most cases 6 points are enough to capture asymmetries in the crests and troughs of a wave. For spectral modelling, this number should be multiplied by 2 to prevent aliasing: only wave numbers below the Nyquest value should be taken into account.

This leads to the problem to find a good estimate of the MRW. We distinguish between harmonic and irregular waves, between flat and varying bottom, and linear and nonlinear simulations. In the first instance for 1D (long crested) waves in the direction of propagation and then some remarks about short crested waves.

Waves above flat bottom

- For harmonic waves we can take MRW as the wave length of the wave, also in case of mildly nonlinear Stokes waves.
- For compositions of a few harmonic waves for linear simulations, MRW is the smallest wave length. For m-th order nonlinear simulations a fraction $1/m$ is

needed but maybe smaller for strongly nonlinear cases; for breaking waves a further reduction may be required.

- For irregular waves and linear simulations, the spectrum will be decisive. Decide about the largest frequency for which the simulation should be accurate (for instance 2, 3 or 4 times the peak frequency if the spectrum is not too broad), and take the corresponding wave length as MRW. Observe that as a consequence of the concavity of the dispersion relation $\omega = \Omega(k)$, 2 times larger frequency implies a much larger than 2 times larger wave number (much smaller than two times smaller wave length.)
- Nonlinear (breaking) simulations of irregular waves can be treated in the same way as described above for combinations of harmonic waves.

Waves over varying bottom

Bathymetric changes can, just as nonlinear effects, strongly deform (and extend) the spectrum, for instance by the process of mode generation. Clearly, in such cases the gridsize should be fine enough to capture all details, until a further refinement does not show essential differences anymore.

For *short crested waves* there will usually be a 'main' direction of propagation, although that direction does not have to coincide with the direction of main energy propagation (which is the direction that causes the main changes in the wave profiles) in multi-modal sea states. Then the above arguments for long crested waves hold in that case in the propagation direction. But since interesting cases of short crested waves will usually have a spreading angle of some 30^0 or more with respect to the main direction, there is not much to gain for the gridsize in the perpendicular direction, so that a square or unstructured grid is often most efficient and accurate.

Practical Do's and Don'ts (in HAWASSI software):

We list some practical points that may help to perform simulations with HAWASSI in a smooth way. Adding to the manuals, the following points should be taken into account.

- Specify in the MAIN GUI an appealing Project name, and use User's note to indicate special cases, or changes; all these can be found back in the log-file.
- Check the provided input after finishing the 'Preparation' phase, but before running the software; check the domain and bathymetry, verify the input time signal, the quality of dispersion, etc.
- Get a good quantitative idea about the value of essential quantities: what will be the peak wave length, period, the expected Hs, Hs/Depth, and qualitative aspects such as shoaling, role of non-linearity, etc.
- In the Post-Processing phase, a simulation may easily show if something went wrong or that it seems to be correct at first sight. Then look at time signals (wave profiles) and their spectra at different places (times), to convince yourself that the result can be trusted or not. Do not just click and plot but look and study graphical output intensely.
- Finally contemplate on (dis-)agreement of the simulation results and your anticipated results. Based on these results you should decide about (changes, extensions for) next simulations

APPENDIX B

Variational calculus

We provide a short description of the basic concepts, definitions and techniques.

1. Basic concepts and definitions

1.1. Fermat. In real analysis courses at an introductory level, functions of one or more variables are considered. The definition of differentiation of functions is a vital part of such courses, and a standard result is the following.

> **Algorithm of Fermat**, for 1-D optimization problems[1].
>
> If the differentiable scalar function of one variable $f : \mathcal{R} \to \mathcal{R}$ attains a (local) extreme value at the point $\hat{x}$, then the derivative at that point vanishes:
>
> $$f'(\hat{x}) = 0.$$

Viewed as a condition for a point to be an extremal element, this condition is necessary but not sufficient; every point that satisfies this property is called a *critical point*, so including 'saddle points'.

Knowing the above result for functions of one variable, the generalization to functions of more (n) variables is remarkably simple: the use of partial derivatives reduces the n dimensional problem to n 1-D problems, as follows.

For $F : \mathcal{R}^n \to \mathcal{R}$, at the point x the *directional derivative* in a direction η is found by differentiating the scalar function obtained by restricting F to the line through x in the direction η, i.e. the function $\varepsilon \to F(x + \varepsilon\eta)$,

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(x + \varepsilon\eta) \equiv DF(x)\eta.$$

Here $DF(x)$ is seen as a map from $\mathcal{R}^n$ into $\mathcal{R}$ as $\eta \to DF(x)\eta$. If x minimizes F on $\mathcal{R}^n$, this point certainly minimizes at $\varepsilon = 0$ the restriction to the line, and hence

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} F(x + \varepsilon\eta) = DF(x)\eta = 0.$$

If x minimizes F on $\mathcal{R}^n$, this should hold for any direction η, and the vanishing of the directional derivative in every direction η can be expressed by writing

$$DF(x) = 0.$$

It is common to rewrite this property by using the notion of gradient as follows.

[1]Fermat did not write down the actual equation; he reasoned that small variations near a minimizer produces a higher order variation in the function, the fundamental idea that leads to the result and justifies to adhere his name to the mathematical algorithm. Fermat didn't know the concept of derivative of functions other than polynomials; it was Leibniz who introduced in 1684 the concept of derivative of arbitrary functions.

For $F : \mathcal{R}^n \to \mathcal{R}$, let ∇F be the *gradient of the function*, defined to be the column vector

$$\nabla F(x) = \begin{pmatrix} \partial_{x_1} F(x) \\ ... \\ \partial_{x_n} F(x) \end{pmatrix}.$$

The relation with the directional derivative is simply that for each η

$$DF(x)\eta \equiv \nabla F(x) \cdot \eta$$

where in the rhs the usual inner product of vectors in $\mathcal{R}^n$ is meant. Then, from $DF(x)\eta = \nabla F(x) \cdot \eta = 0$ for all η, the vanishing of the map $DF(x)$ can now be expressed by the vanishing of the gradient (vector)

$$\nabla F(x) = 0,$$

a system of n coupled equations for the n-vector x. This formulation is the direct generalization of Fermat's algorithm to n dimensional optimization problems.

1.2. Generalization to infinite dimensions. In wave applications we encounter scalar functions defined on an (infinite dimensional) function space. The function is then usually called a function*al*. The function space consists of functions of one or several variables defined in some domain. The functional is often of the form of a *density functional*: it assigns to each function a real number by integrating powers of the function and its derivative: for instance the L_2-norm of a function $u(x)$

$$\|u\|_{L_2} := \sqrt{\int u^2(x)\, dx}.$$

The notion of differentiation of such functionals is a simple generalization of differentiation of functions of a finite number of variables, as follows.

- By restricting the functional to 1-dimensional lines in the function space, the notion of directional derivative can be applied and is called the *first variation*;
- When dealing with density-functionals, a generalization of the gradient can be defined and will lead to the notion of *variational derivative*. The specific expression is related to the choice of the innerproduct, but without further notice we will always use the standard L_2 innerproduct for functions under consideration, with notation:

$$< f, g >:= \int f(x)\, .g(x)\, dx$$

- The fact that from $\nabla F(x) \cdot \eta = 0$ for all η, it follows that $\nabla F(x) = 0$, can be generalized to infinite dimensional function spaces with the L_2 innerproduct as a consequence of *Lagrange's Lemma*; this result will enable us to identify the first variation with the variational derivative (not considering boundary contributions).

The infinite dimensional case also brings characteristic differences.

- The functions to which the functional assigns a certain value are defined on a domain; 'variations' of the functions may be restricted to variations in the interior of the domain because of restrictions on the boundary by so-called boundary conditions; but such conditions may also result as a consequence of the condition for a critical point (so-called natural boundary conditions).

- When talking about functions, clearly their smoothness (continuity, differentiability) is important[2].

The typical notation to be used in the following for the variational derivative is $\delta\mathcal{L}(u)$, and Fermat's algorithm generalizes to

$$\delta\mathcal{L}(u) = 0 \tag{B.1}$$

as the condition for a minimizing element. This equation is most times a differential equation together with boundary conditions, replacing the algebraic equation $\nabla F(x) = 0$ for a minimizer of a function of a finite number of variables.

Just as in finite dimensions, the second derivative may reflect minimization properties, and in general provide insight into the character of a critical point. In the Calculus of Variations these aspect are dealt with in the *theory of first and second variation*.

1.3. Notation and General Formulation. Generally speaking, for an optimization problem we have the following basic ingredients:

- a set of admissible elements $\mathcal{M}$, usually some subset of an (infinite dimensional) space $\mathcal{U}$;
- a functional $\mathcal{L}$, defined on $\mathcal{U}$ (or only on $\mathcal{M}$).

The minimization problem of $\mathcal{L}$ on $\mathcal{M}$ concerns questions about an element $\hat{u}$ that minimizes the functional on the set of admissible elements. By definition $\hat{u}$ is the element for which the functional achieves it's lowest value μ among all admissible elements in the set $\mathcal{M}$

$$\mu = \mathcal{L}(\hat{u}) \text{ and } \mathcal{L}(\hat{u}) \leq \mathcal{L}(u) \quad \text{for all} \quad u \in \mathcal{M}.$$

The questions could deal with the existence, the uniqueness, and the characterization and computation of the minimizer. Here we will mainly deal with the equation as the *characterization* of a minimizer and critical points; these equation(s) for a critical point are called the Euler-Lagrange equation, and may be supplemented with boundary conditions, etc..

1.4. Functionals. The functionals we will encounter most are *density-functionals*: the functional assigns to each function (from a set of admissible functions) a number that is found by integrating expressions of the function and its derivatives. The general form of a density functional cannot be given without complicated notation; therefore we just list the main type of functionals that we encounter in variational wave modelling.

1.4.1. *Lagrangian functionals from Classical Mechanics.* For scalar or vector functions q, say $q \in R^N$, of one variable t (the time), and with $\dot{q}$ denoting the time derivative of q, the functional is of the general form

$$\mathcal{L}(q) = \int L(q, \dot{q})dt \tag{B.2}$$

where the Lagrangian density L is a given, smooth, function of its $2N$ ($R^N \times R^N$) arguments.

[2]It also leads to the most characteristic difference with finite dimensional spaces. In infinite dimensional spaces different (non-equivalent) norms are possible: a function may be square integrable, while the squared derivative may have arbitrary large integral.

1.4.2. *Dirichlet type of integrals.* For scalar functions Φ defined on a domain $\Omega \subset R^n$, and with $\nabla\Phi$ the gradient of Φ, the functional integrates over the spatial domain an expression, like

$$\mathcal{L}(\Phi) = \int_\Omega \left[|\nabla\Phi|^2 - \rho(x)\Phi\right] dx$$

where ρ is a given function on Ω.

1.5. Spaces and Admissible variations. For each variational problem a set of admissible elements should be specified: the set on which the functional is defined, and the functions that are allowed ('admissible') in the competition of looking for the minimizer.

In general, the *set of admissible elements* consists of functions $u(x)$ of one or more variables $x \in R^n$ defined on a certain domain (bounded or not) $\Omega \subset R^n$. Usually these functions are subject to certain conditions, conditions which may be of different character

- : *smoothness and integrability conditions*, at least to assure that the density functional is well defined;
- : *boundary conditions*: conditions on the function and/or its derivatives on (parts of) the boundary $\partial\Omega$ of the domain Ω on which the functions are defined;
- : (internal) *constraints*; since this is not relevant for most wave problems, we will not dwell upon constrained problems.

To explain the notion of (admissible) variation, consider a function $u(x)$ defined on Ω. A *variation* of that function is a 'small' change of that function, possibly over its full domain Ω. To make 'small' more precise: for any (finite) function $\eta(x)$ on Ω and ε sufficiently small, the function $\varepsilon\eta(x)$ is small and we can consider the function $u(x) + \varepsilon\eta(x)$ to be in the 'neighbourhood' of $u(x)$, approaching $u\,(x)$ for $\varepsilon \to 0$. (Note: we do not specify a certain topology and hence not a precise notion of 'approach'.) Stated differently, a line through the 'point' u in the 'direction' η is the set in function space:

$$\varepsilon \to u(x) + \varepsilon\eta(x);$$

it is a family of graphs in which the original graph of $u(x)$ is embedded, and the elements of the family 'approach' this function for vanishing ε.

In the classical Calculus of Variations it is common to write $\delta u(x)$ for $\varepsilon\eta(x)$ with ε small and to call it a variation; to avoid confusion, we will often write $\varepsilon\eta(x)$ and make the dependence on the parameter ε explicit.

In all the relevant wave applications, we will deal with admissible sets for which each element (function) can be changed locally in a neighbourhood of each *interior* point in its domain of definition in an arbitrary way and that still the varied element remains admissible.

To consider such local variations, we introduce test functions. Test functions make it possible to change a given function in the interior of its domain of definition, without altering the behaviour at the boundary.

DEFINITION 1. *Given a domain $\Omega \subset R^n$, the set of* test functions *on Ω will be denoted by $C_0^\infty(\Omega)$ and consist of all functions that are infinitely differentiable (C^∞) and that vanish, together with all derivatives, near the boundary $\partial\Omega$ (C_0^∞).*

REMARK 10. *Such test functions really exist: for any interior point* $x_0 \in \Omega$ *and* r_0 *such that* $x \in \Omega$ *if* $|x - x_0| < r_0$ *an example is*

$$\eta(x) = \begin{cases} \exp\left(-\frac{1}{r_0^2 - |x-x_0|^2}\right) & \text{for } |x - x_0| < r_0 \\ 0 & \text{for } |x - x_0| \geq r_0 \end{cases}$$

Now we can define two essentially different classes of admissible elements, leading to different variational problems and different methods and results:

DEFINITION 2. *An* unconstrained variational problem *is a problem for which the set of admissible elements* $\mathcal{M}$ *consists of functions defined on a domain* Ω *such that all test functions belong to the set of admissible variations:*

$$\text{if } u \in \mathcal{M} \quad \text{then } u + C_0^\infty(\Omega) \subset \mathcal{M} \ ,$$

meaning $u + \varepsilon\eta \in \mathcal{M}$ *for each* $\eta \in C_0^\infty(\Omega)$ *and each* ε.
When this is not the case we will talk about a constrained variational problem.

We will deal only with unconstrained problems: $\mathcal{M}$ will consists of 'all' smooth functions on Ω but possibly with restrictions on the boundary (boundary conditions).

2. Theory of first variation

In this section we derive the generalization of Fermat's algorithm. Assuming the existence of a minimizer, we derive the governing equation, after which all solutions of this equation are called critical points, just as in finite dimensions.

2.1. First variation and variational derivative. To define the derivative of a functional, we use the idea of directional derivative since then the problem is reduced to the differentiation of a scalar function of only one variable. Hence, let u be a given function, and v an (arbitrary) variation. With this variation the original function u is embedded in the class of varied functions (a 1-parameter family) of the form $\varepsilon \mapsto u+\varepsilon v$. Fixing v, and restricting the functional to this line, we get a scalar function of one variable:

$$\varepsilon \mapsto \mathcal{L}(u+\varepsilon v).$$

The derivative of this function of ε is then by definition the directional derivative, the first variation.

Definition 3. First variation
The first variation of a functional $\mathcal{L}$ at u in the direction v is denoted by $\delta\mathcal{L}(u;v)$ and defined as

$$\delta\mathcal{L}(u;v) = \frac{d}{d\varepsilon}\mathcal{L}(u+\varepsilon v)\bigg|_{\varepsilon=0}. \tag{B.3}$$

In most cases, the first variation is linear in v (nonlinear in u in general). When it is linear in v (and continuous with respect to a topology on the space), it is known as the Gateaux-derivative and is the direct generalization of the directional derivative of a function on a finite dimensional space.

From the definition of first variation above, it follows directly that a linear approximation of $\mathcal{L}(u+\varepsilon v)$ is given as

$$\mathcal{L}(u+\varepsilon v) = \mathcal{L}(u) + \varepsilon\delta\mathcal{L}(u;v) + o(\varepsilon) \tag{B.4}$$

where, here and in the following, $o(\varepsilon)$ means terms that are of higher than first order in ε: $o(\varepsilon)/\varepsilon \mapsto 0$ for $\varepsilon \mapsto 0$.

The definition above applies to all kind of functionals. For density functionals it is usually possible to perform a partial integration and to rewrite $\delta\mathcal{L}(u;v)$ as the $L_2(\Omega)$-innerproduct of v and some function which will be denoted by[3] $\delta\mathcal{L}(u)$ and which will be the direct generalization of the gradient of a function of a finite number of variables. This may require that the function u is smooth enough, and usually a contribution consisting of an integration over the boundary appears in addition:

$$\delta\mathcal{L}(u;v) = \int_\Omega \delta\mathcal{L}(u)\cdot v + \int_{\partial\Omega} b(u;v) \tag{B.5}$$

If functions v are considered that vanish on the boundary, the boundary contribution vanishes identically. Therefore, we can use in particular the class of test functions $C_0^\infty(\Omega)$ to avoid these boundary contributions. Then we have the following notion.

[3]For notational convenience we will exploit the notation $\delta\mathcal{L}(u)$, although in much of the literature the notation $\delta\mathcal{L}/\delta u$ is often used:

$$\delta\mathcal{L}(u) \equiv \frac{\delta\mathcal{L}}{\delta u}(u).$$

DEFINITION 4. *The function $\delta\mathcal{L}(u)$ on Ω defined by the condition*

$$\delta\mathcal{L}(u;\eta) =< \delta\mathcal{L}(u), \eta > \equiv \int_\Omega \delta\mathcal{L}(u)\cdot\eta\ dx,\ \textit{for all } \eta \in C_0^\infty(\Omega) \tag{B.6}$$

is called the variational derivative *of the functional $\mathcal{L}$ at the point u.*

It will follow from Lagrange's Lemma 1 below that when $\delta\mathcal{L}(u)$ is continuous, (B.6) indeed defines the function $\delta\mathcal{L}(u)$ uniquely. We will give examples in the following to demonstrate the calculation of the variational derivative.

2.2. Characteristic cases. For the characteristic functionals mentioned before, the first variation and the variational derivative will be given here in illustrative notation.

2.2.1. *Lagrangian functionals from Classical Mechanics.* For

$$\mathcal{L}(q) = \int L(q,\dot{q},t)dt$$

(using vector notation, and a sloppy, but characteristic, way of writing the derivative of L with respect to the 'variables') the first variation and variational derivative are given by

$$\begin{aligned}\delta\mathcal{L}(q;\xi) &= \int\left[\frac{\partial L}{\partial\dot{q}}\dot{\xi} + \frac{\partial L}{\partial q}\xi\right]dt,\\ \delta\mathcal{L}(q) &= -\frac{d}{dt}\left[\frac{\partial L}{\partial\dot{q}}\right] + \frac{\partial L}{\partial q}.\end{aligned} \tag{B.7}$$

2.2.2. *Dirichlet type of integrals.* For the functional

$$\mathcal{L}(\Phi) = \int_\Omega\left[\frac{1}{2}|\nabla\Phi|^2 - \rho(x)\Phi\right]dx$$

the first variation is given by

$$\delta\mathcal{L}(\Phi,\psi) = \int_\Omega\left[\nabla\Phi\cdot\nabla\psi - \rho(x)\psi\right]dx$$

To find the variational derivative we have to perform a 'partial integration' for this multiple integral. This is done by using the following basic elements from differential calculus. We write for a vector $\mathbf{a}$ it's divergence as $\mathrm{div}(\mathbf{a})$ or as $\nabla\cdot\mathbf{a} = \mathrm{div}(\mathbf{a})$). Then we have the basic identity:

$$\text{for scalar } \alpha \text{ and vector } \mathbf{a}: \qquad \mathrm{div}(\alpha\mathbf{a}) = \nabla\alpha\cdot\mathbf{a} + \alpha\mathrm{div}(\mathbf{a}) \tag{B.8}$$

with which we can write $\nabla\Phi\cdot\nabla\psi = \mathrm{div}(\psi\nabla\Phi) - \psi\mathrm{div}(\nabla\Phi)$.
Secondly we recall

$$\textbf{Gausz' theorem}: \int_\Omega \mathrm{div}(\mathbf{a}) = \int_{\partial\Omega}\mathbf{a}\cdot\mathbf{n} \tag{B.9}$$

where $\mathbf{n}$ is the outward pointing normal to the boundary $\partial\Omega$ of Ω. Then we find:

$$\int_\Omega \nabla\Phi\cdot\nabla\psi = \int_\Omega -\psi\nabla\cdot\nabla\Phi + \int_{\partial\Omega}\psi\nabla\Phi\cdot\mathbf{n}$$

Using the common notation: $\partial_n\Phi = \nabla\Phi\cdot\mathbf{n}$ for this *normal derivative*, we have found

$$\delta\mathcal{L}(\Phi,\psi) = \int_\Omega\left[-\nabla\cdot\nabla\Phi - \rho(x)\right]\psi dx + \int_{\partial\Omega}\psi\partial_n\Phi \tag{B.10}$$

To find the variational derivative, we restrict to test functions (which vanish at the boundary) and find

$$\delta\mathcal{L}(\Phi) = -\nabla\cdot\nabla\Phi - \rho(x) = -\Delta\Phi - \rho\,(x) \tag{B.11}$$

where we recall the special notation for the Laplace operator:

$$\textbf{Laplace operator}: \quad \Delta = \nabla\cdot\nabla = \partial_x^2 + \partial_y^2 + ...$$

so, for instance

$$\delta\int_\Omega \frac{1}{2}|\nabla\Phi|^2 = -\Delta\Phi$$

2.3. Stationarity condition. We now consider the basic optimization problem. Considering $\mathcal{L}(u+\varepsilon v)$ for an admissible variation, in general this value will differ from $\mathcal{L}(u)$ in first order in ε as follows from (B.4). A critical point will be defined by the fact that this difference is of higher (most times second) order.

Definition 5. *A point $\hat{u}$ is called a* critical point, or stationary point*, of the functional $\mathcal{L}$ on the set $\mathcal{M}$ if the following holds:*

$$\delta\mathcal{L}(\hat{u}; v) = 0 \quad \textit{for all admissible variations.} \tag{B.12}$$

As in finite dimensions, the notion of 'critical point' is a generalization of a local maximum or minimum.

Proposition 1. *If $\mathcal{L}$ has a local maximal or minimal value at $\bar{u}$, then $\bar{u}$ is a critical point of $\mathcal{L}$.*

This is a basic result in the theory of 'first variation': *(B.12) gives the condition for a point to be a critical point, and this is a necessary condition (usually not sufficient) for a point to be a local maximum or minimum.*

2.4. Euler-Lagrange equation. It is possible to translate condition (B.12) into an explicit equation for $\hat{u}$ along the following lines.

Let $\hat{u}$ be a critical point of the *unconstrained* variational problem for $\mathcal{L}$ on $\mathcal{M}$. From the stationarity condition (B.12) and the fact that all test functions are admissible variations, it follows that certainly it must hold that

$$\delta\mathcal{L}(\hat{u}; \eta) \equiv \langle \delta\mathcal{L}(\hat{u}), \eta\rangle = 0 \quad \text{for all } \eta \in C_0^\infty(\Omega). \tag{B.13}$$

This leads to the equation for $\hat{u}$:

Proposition 2. Euler-Lagrange equation
If $\hat{u}$ is a critical point of the variational problem for $\mathcal{L}$ on $\mathcal{M}$, then (provided $\delta\mathcal{L}(\hat{u})$ is a continuous function) $\hat{u}$ satisfies

$$\delta\mathcal{L}(\hat{u}) = 0. \tag{B.14}$$

This equation for $\hat{u}$ is called the Euler-Lagrange equation of the functional $\mathcal{L}$.

The proof of this result is an immediate consequence of the first order condition (B.13) and the following basic Lemma.

Lemma 1. Lagrange's Lemma
Let f be a continuous function on Ω that is such that

$$\int_\Omega f(x)\eta(x)dx = 0 \textit{ for all } \eta \in C_0^\infty(\Omega).$$

Then f vanishes identically on (the interior of) Ω: $f(x) = 0$ for all $x \in \Omega$.

PROOF. Suppose that at some interior point x^* of Ω the function f does not vanish, say has value $\alpha > 0$. Then, from continuity of f, there is a small neighbourhood of f such that f doesn't vanish there, and in fact, $f(x) > \alpha/2$ for all $x, |x - x^*| < r_0$ for small enough r_0. Now take a test function, say $\bar{\eta}$, that is nonnegative (positive) inside, and vanishes outside this neighbourhood. Then $\int_\Omega f(x)\bar{\eta}(x)dx > \alpha/2 \int_\Omega \bar{\eta}(x)dx > 0$, contradicting the assumption. □

2.5. Natural boundary conditions. From the vanishing of the first variation for all test functions, the Euler-Lagrange equation is obtained. But there may be more admissible variations than only test functions. In that case, for a critical point it should also hold that the boundary contribution in (B.5) vanishes:

$$\int_{\partial\Omega} b(\hat{u}; v) = 0, \quad \text{for all admissible variations.} \tag{B.15}$$

For admissible variations different from test functions this condition will give certain conditions for $\hat{u}$ on the boundary $\partial\Omega$. If this happens, these conditions are called *natural boundary conditions*: they appear as additional conditions for a critical point, not by the requirement that $\hat{u}$ should belong to $\mathcal{M}$, but from (B.15), which is a consequence of the stationarity condition (B.12). We will give an example in the next section.

3. Principle of Minimal Potential Energy

For time independent problems, or for stationary states of time dependent problems, the actual physical state may be described by a *principle of minimum (potential) energy*, which means the following:

- there is a set of admissible, physically acceptable, states $\mathcal{M}$,
- there is a (potential) energy functional $\mathcal{D}$ that assigns a value ("energy-like") $\mathcal{D}(u)$ to each state $u \in \mathcal{M}$,
- the actual physical state is the state $\hat{u}$ that minimizes $\mathcal{D}$ on $\mathcal{M}$.

We present some examples to illustrate the applicability.

3.1. Dirichlet's principle. Consider a domain $\Omega \subset \mathcal{R}^3$ with boundary that consists of two parts $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. We are interested to find the stationary irrotational flow of an incompressible fluid, when at part of the boundary $\partial\Omega_1$ the fluid is inserted with a known normal speed u_1.

To proceed, we introduce the fluid potential Φ such that the fluid velocity is given by $\mathbf{u} = \nabla\Phi$, and consider the functional (note the boundary integral!)

$$\mathcal{D}(\Phi) = \int_\Omega \frac{1}{2}|\nabla\Phi|^2 dx - \int_{\partial\Omega_1} u_1(x)\Phi d\sigma.$$

Dirichlet's *principle* states that if the actual field exist, it is such that it minimizes this functional among all potentials. The minimization problem

$$\hat{\Phi} \in \text{Min } \mathcal{D}(\Phi)$$

leads to the vanishing of the first variation

$$\delta\mathcal{D}(\hat{\Phi}, \psi) = \int_\Omega \nabla\hat{\Phi}.\nabla\psi dx - \int_{\partial\Omega_1} u_1(x)\psi d\sigma = 0$$

which after applying Gausz theorem leads to:

$$\delta\mathcal{D}(\hat{\Phi}, \psi) = \int_\Omega -\Delta\hat{\Phi}.\psi dx + \int_{\partial\Omega} \partial_n\hat{\Phi}.\psi - \int_{\partial\Omega_1} u_1(x)\psi d\sigma = 0.$$

Restricting first to test functions ψ, the vanishing at the boundary leads to the Euler-Lagrange equation $-\Delta\hat{\Phi} = 0$ in the interior, so that there remains

$$\int_{\partial\Omega_2} \partial_n\hat{\Phi}.\psi + \int_{\partial\Omega_1} \left[\partial_n\hat{\Phi} - u_1(x)\right]\psi d\sigma = 0.$$

Now, variations of ψ at the boundary part $\partial\Omega_1$ that vanish at $\partial\Omega_2$ lead to the required influx condition, while arbitrary variations of ψ at the boundary part $\partial\Omega_2$ that vanish at $\partial\Omega_1$ lead to vanishing of the normal component at $\partial\Omega_2$, and we get the Neumann boundary value problem:

$$\begin{cases} -\Delta\hat{\Phi} &= 0 \quad \text{in } \Omega, \\ \partial_n\hat{\Phi} &= u_1 \quad \text{on } \partial\Omega_1, \\ \partial_n\hat{\Phi} &= 0 \quad \text{on } \partial\Omega_2. \end{cases}$$

Observe that the Neumann condition on $\partial\Omega_2$ arises as a natural boundary condition, and that the boundary term in the integral produces the correct influx condition at $\partial\Omega_1$.

However, the above is purely formal, and we have no guarantee that a solution of this problem actually exist. In this case, physical intuition can easily predict that most times such a stationary solution will NOT exist: only inflow (or outflow) at the

boundary $\partial\Omega_1$ and no in- or outflow through the remaining boundary $\partial\Omega_2$ will in general conflict with the incompressibility condition. We can expect a solution to exist only if a compatibility condition is satisfied. This condition results in this simple case from integrating the Euler equation and applying Gausz theorem:

$$\int_\Omega -\Delta\Phi = 0 = \int_{\partial\Omega} \partial_n \Phi d\sigma.$$

Hence a necessary condition for the given boundary problem to have a solution is that

$$\int_{\partial\Omega_1} u_1(x)\, d\sigma = 0,$$

i.e. that there is no nett flow into the domain, confirming the physical intuition. It can actually be proved that this necessary condition is also sufficient: for total zero influx, there exist a stationary flow.

EXERCISE 12. (1) *Show that a critical point of*

$$Crit\ \{\frac{1}{2}\int_\Omega \left[|\nabla\Phi|^2 - \rho(x)\,\Phi\right] dx - \int_{\partial\Omega_2} \psi_2\Phi \mid \Phi(x) = \psi_1(x) \text{ for } x \in \partial\Omega_1\ \}$$

satisfies

$$\begin{cases} -\Delta\Phi &= \rho & \text{in } \Omega, \\ \Phi &= \psi_1 & \text{on } \partial\Omega_1, \\ \partial_n\Phi &= \psi_2 & \text{on } \partial\Omega_2. \end{cases}$$

(2) *Show that there exists at most one critical point, and that, if it exists, it is in fact a minimizer.*

(3) *When $\partial\Omega_1$ is empty, derive the necessary condition between ψ_2 and ρ for a solution to exist. How is this condition related to the finiteness of the minimum value, i.e. to the boundedness from below of the functional?*

3.2. Weak formulation. We have seen that the variational derivative of a density functional follows from the first variation by applying a partial integration, restricted to test functions to avoid contributions at the boundary:

$$\delta\mathcal{L}(\hat{u};\eta) \equiv \langle \delta\mathcal{L}(\hat{u}), \eta\rangle \quad \text{for all } \eta \in C_0^\infty(\Omega). \tag{B.16}$$

Without having explicitly stated, when writing down the Euler-Lagrange equation $\delta L(u) = 0$, in general some smoothness has to be assumed for the extremal function.

On the other hand, we have seen that the first variation is the basic result from differentiation in a direction, and so without partial integration that result is valid. So we could just as well *interpret* the result after partial integration even in cases when it is not a continuous function, by requiring (B.16) to hold! This turns out to be a very fruitful idea to generalize the notion of differentiability of functions that are not differentiable in the classical way. This is then often called generalized derivative, or distributional derivative (in the theory of generalized functions). When we interpret a BVP in this way, we often talk about the variational formulation of the problem, or about the *weak formulation*. It is also the basis of Finite Element methods, where a given equation is not interpreted pointwise, but integrated against spline functions, so that a weak formulation results.

4. Dynamical Systems and Evolution Equations

In this section we consider dynamical systems with a variational structure, systems from Classical Mechanics and some problems from wave dynamics.

4.1. Classical Mechanics. Problems from Classical Mechanics deal with the motion of a finite number of point-masses, usually with interactions between them and in a force field. These systems are described by ordinary differential equations for the position of each of the masses. Roughly speaking, these equations are in the form of Newton's law of force. But, and that is characteristic for Classical Mechanics, the systems are in general 'conservative', there is no dissipation, and the total energy is conserved. From the conservative nature it follows that the equations of motion themselves have a variational structure: the actual motions are critical points of a certain functional, the 'action functional'. We consider two ways of describing such systems; the Lagrangian and the Hamiltonian way. It may be observed that the integrals are integrals over time, from an initial to a final time. Boundary conditions are such that at these moments the actual configuration is supposed to be known; this boundary-value formulation is very different from an initial-value problem, but is needed to fit in the variational frame.

4.1.1. *Lagrangian systems.* First the general definition, with the characteristic nomenclature, then simple examples

DEFINITION 6. *A dynamical system with position vector $q \in R^N$ is a Lagrangian system if a Lagrangian $L(q, \dot{q}, t)$ can be given such that critical points of the* action functional *(or* Lagrangian functional)

$$\mathcal{L}(q) = \int L(q, \dot{q}, t)dt$$

(that satisfy the corresponding Euler Lagrange equations) are the dynamic equations of the system. This variational principle is often called the action principle.

Consider the motion of a single mass-point of mass m moving along the x-axis, position at time t denoted by $q(t)$ and the Lagrangian functional

$$L(q) = \int \left[\frac{1}{2}m\dot{q}^2 - V(q,t)\right] dt$$

which assigns to a certain trajectory $t \to q(t)$ between initial and final time the value determined by L.

The action-principle then states that the actual, physical trajectory is the one that is a critical point of the Lagrangian functional; more precisely; given initial position $q(t_i) = P$ and final position $q(t_f) = Q$, the admissible trajectories are those that connect these points in the specified time interval and the admissible variations are deformations of the trajectory that vanish at the initial and final time. The Euler-Lagrange equations is given by:

$$m\ddot{q} + \frac{\partial V}{\partial q} = 0.$$

This is a simple form of *Newton's equation* for a system of one degree of freedom: $\ddot{q}$ is the acceleration, and $-dV/dq$ is the 'force', which is here the derivative of the so-called *potential energy* function $V(q)$; such force-fields are called 'conservative'.

Note that the *Lagrangian density is the difference between kinetic energy and potential energy*; this is typically the case for systems from Classical Mechanics..

Also, consider the *total energy* $E(q,\dot{q})$:

$$E(q,\dot{q}) := \frac{1}{2}m\dot{q}^2 + V(q,t)$$

and calculate directly that for solutions of the equations it holds

$$\frac{d}{dt}E(q,\dot{q}) = m\ddot{q}\dot{q} + \frac{\partial V}{\partial q}\dot{q} + \frac{\partial V}{\partial t} = \left[m\ddot{q} + \frac{\partial V}{\partial q}\right]\dot{q} + \frac{\partial V}{\partial t} = \frac{\partial V}{\partial t}.$$

Hence, *if the Lagrangian density does not explicitly depend on time, i.e.* $\frac{\partial V}{\partial t} = 0$*, the total energy is conserved*:

$$\frac{d}{dt}E(q,\dot{q}) = 0 \text{ if } \frac{\partial V}{\partial t} = 0.$$

Again, this is a property that holds in more general systems as well, as we shall see.

For systems of one degree of freedom, energy conservation allows phase-plane analysis: in the phase plane $(q,\dot{q})$ the motion of the particle is restricted to a level set of the total energy $E(q,\dot{q}) = E_0$, where the value E_0 is determined by specifying an initial position and velocity.

Exercise 13. *Consider as a specific example of an* infinite dimensional Lagrangian system *for functions* $u(x,t)$ *the Lagrangian density*

$$L(u,\partial_t u) = \int \left[\frac{1}{2}(\partial_t u)^2 - \frac{c^2}{2}(\partial_x u)^2 - fu\right] dx;$$

note the notation: $L(u,\partial_t u)$ *is, on the one hand, a functional as far it concerns the dependence on* x*, and therefore it actually depends on two functions, here denoted by* u *and* $\partial_t u$*, which makes sense after considering the action functional which really maps the function* u *of* (x,t) *into the reals:*

$$\int L(u,\partial_t u)dt$$

The Euler-Lagrange equation is a forced wave equation

$$\partial_t^2 u = c^2\partial_x^2 u - f$$

which is supplemented by prescribed and/or natural boundary conditions, depending on the conditions of the functions in spatial variables. A more dimensional analog, with more spatial dimensions, is

$$L(u,\partial_t u) = \int \left[\frac{1}{2}(\partial_t u)^2 - \frac{c^2}{2}|\nabla u|^2 - fu\right] dx$$

with Euler-Lagrange equation

$$\partial_t^2 u = c^2\Delta u - f$$

4.1.2. *Classical Hamiltonian systems.* One way to introduce Hamiltonian systems is as an alternative description of Lagrangian systems, although the correspondence is not one-to-one. The observation is that the Euler Lagrange equations for Lagrangian L are typically second order in time. If a first order in time description is preferred (for which there may good reasons, for instance from a conceptual point of view of the IVP – Initial Value problem –) this can be done by introducing more dependent variables: except the position vector $q \in R^N$ one introduces momentum-type of variables $p \in R^N$ and looks for first order in time system of equations in the pair $(q,p) \in R^N \times R^N$. Starting with a Lagrangian system, this can often be done in a systematic way using Legendre transformation.

Somewhat more general, we define

DEFINITION 7. *A dynamical system is called a classical* Hamiltonian system *if the dynamical equations can be described with pairs of variables* $(q,p) \in R^N \times R^N$ *in the so-called* phase space*, and with a* Hamiltonian $H(t,q,p) : \mathcal{R} \times R^N \times R^N \to \mathcal{R}$ *in such a way that the dynamical equations are found from the* canonical action principle*, which means as the critical points of the* canonical action functional

$$\mathcal{A}_c(q,p) = \int \left[p(t) \cdot \partial_t q(t) - H(t,q(t),p(t))\right] dt,$$

and hence satisfying the so-called Hamilton equations

$$\left\{ \begin{array}{lcl} \partial_t q & = & \frac{\partial H}{\partial p} \\ \partial_t p & = & -\frac{\partial H}{\partial q} \end{array} \right. \tag{B.17}$$

In many problems from classical and continuous mechanics, *the Hamiltonian is the sum of kinetic and potential energy, i.e. the total energy*; this is different from the Lagrangian, which is the difference between kinetic and potential energy.

Almost immediately seen from the equations is that for autonomous Hamiltonians there is *energy-conservation*:

$$\text{when } \frac{\partial H}{\partial t} = 0 \text{ the Hamiltonian is conserved: } \frac{d}{dt} H(q,p) = 0.$$

EXERCISE 14. *Consider as a specific example of an* infinite dimensional Hamiltonian system *for functions* $u(x,t), p(x,t)$ *the Hamiltonian*

$$H(u,p) = \int \left[\frac{1}{2} p^2 + \frac{c^2}{2} (\partial_x u)^2 + fu \right] dx;$$

The Hamiltonian is a functional on the space of spatially depending pairs of functions (u,p)*. The canonical action functional now reads*

$$\int \left\{ \int \left[p \partial_t u \right] dx - H(u,p) \right\} dt$$

and Hamilton's equations are

$$\begin{aligned} \partial_t u &= \delta_p H(u,p) = p \\ \partial_t p &= -\delta_u H(u) = c^2 \partial_x^2 u - f \end{aligned}$$

describing the same forced wave equation as treated earlier in the Lagrangian setting.

4.2. Evolution equations (Nonlinear Wave equations). We now briefly describe the particular structure that is found for various types of non-linear wave equations.

4.2.1. *Boussinesq Equations.* Equations for surface waves on a layer of fluid that depend on one spatial variable but allowing waves running in both directions have the form

$$\partial_t \, u = -\partial_x \, \delta_\eta \, H(u,\eta), \qquad \partial_t \, \eta = -\partial_x \, \delta_u \, H(u,\eta),$$

i.e. written in characteristic form:

$$\partial_t \begin{pmatrix} u \\ \eta \end{pmatrix} = - \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} \delta_u \, H(u,\eta) \\ \delta_\eta \, H(u,\eta) \end{pmatrix}$$

where u is a velocity-type of variable, and η is the surface elevation. The Hamiltonian usually consists of a part that determines the linear dispersive properties, and a part to account for the nonlinearity in the equations.

4.2.2. *KdV (Korteweg - de Vries) Equation.* Equations for surface waves on a layer of fluid, and optical pulses, that depend on one spatial variable and are restricted to waves running mainly in one direction have the form

$$\partial_t \eta = -\partial_x \delta_\eta H(\eta),$$

where η is the surface elevation. Again, the Hamiltonian usually consists of a part that determines linear dispersive properties, and a part to account for the nonlinearity in the equations. The classical Korteweg- de Vries equation (KdV) is of this form for a Hamiltonian given here in physical variables (depending on scaling the coefficients can be made different)

$$H(\eta) = c_0 \int \left[\frac{1}{2}\eta^2 - \frac{D^2}{12} (\partial_x \eta)^2 + \frac{1}{4D}\eta^3 \right] dx$$

so that the equation reads

$$\partial_t \eta = -c_0 \partial_x \left[\eta + \frac{D^2}{6}\partial_x^2 \eta + \frac{3}{4D}\eta^2 \right] \tag{B.18}$$

4.2.3. *NLS (Non-Linear Schrodinger) Equation.* When modulations of a linear monochromatic wave are studied for KdV-type of equations, the complex amplitude $A(x,t)$ satisfies an NLS-type of equation that is of the form of a complex infinite dimensional Hamiltonian system:

$$\partial_t A = i\delta H(A),$$

with Hamiltonian that accounts for linear dispersive and nonlinear effects:

$$H(A) = \int \left[\frac{\beta}{2} |\partial_x A|^2 + \frac{\gamma}{4} |A|^4 \right] dx$$

5. Exercises

(1) *Boussinesq type of equations*
Surface waves (in one horizontal direction x) that decay at infinity ($|x| \mapsto \pm\infty$) can be described in terms of the wave height $\eta(x,t)$ and a velocity $u(x,t)$ in the following form (a Hamiltonian system):

$$\begin{aligned} \partial_t\, u &= -\partial_x \delta_\eta\, H(u,\eta), \\ \partial_t\, \eta &= -\partial_x\, \delta_u\, H(u,\eta). \end{aligned} \tag{B.19}$$

for a suitable functional (the Hamiltonian) $H(u,\eta)$.

(a) Describe the equations in full detail when the Hamiltonian is given by the following functional

$$H(u,\eta) = \int \{\, \frac{1}{2}\, g\eta^2 + \frac{1}{2}\, (u^2 - \frac{1}{3}\, u_x^2)\,\}\, dx.$$

(This set of equations are the 'linearized' equations.)

(b) In another case (shallow water, no dispersion, but nonlinear), the equations are of the form

$$\begin{aligned} \partial_t\, u &= -\partial_x\, \{g\,\eta + \frac{1}{2} u^2\,\}, \\ \partial_t \eta &= -\partial_x\, \{\, u + \beta\, \eta\, u\,\}, \end{aligned}$$

where β is a constant. Determine the value of β such that this system of equations is a Hamiltonian system of the form (3, 4) given above.

(c) Show that the equations have the horizontal momentum as constant of the motion:

$$\int u(x)\eta(x)dx$$

(2) The *KdV-eqn* in normalized form is given by

$$\partial_t u = -\partial_x \left[u + \partial_x^2 u + u^2\right].$$

(a) Show that it can be written as a generalized Hamiltonian system by determining the Hamiltonian H such that

$$\partial_t\, u(t) = \partial_x\, \delta H_{KdV}(u)$$

(b) Show that the following functionals are constants of the motion for KdV:

$$\int u(x)\, dx, \int u(x)^2\, dx, \text{ and } H_{KdV}(u).$$

(3) The *BBM-eqn* in normalized form is given by

$$\partial_t\, u - \partial_t \partial_x^2 u = -\partial_x (u + \frac{1}{2} u^2).$$

(a) Show that it can be written as a Hamiltonian system by determining the Hamiltonian H and L that is the inverse of a suitable differential operator such that it is given by

$$\partial_t\, u(t) = L\, \partial_x\, \delta H_{BBM}(u)$$

(b) Show that the following functionals are constants of the motion for BBM:

$$\int u(x)\, dx, \int u(x)^2\, dx, \text{ and } H_{BBM}(u).$$

(4) Show that *Burgers' eqn*

$$\partial_t u + u\partial_x u = \partial_x^2 u$$

can be written as a combination of a conservative and dissipative structure by determining functionals D and H such that it gets the form

$$\partial_t\, u = \partial_x\, \delta H(u) + \delta D(u)$$

(5) ** *Variations of the boundary*
Consider (for simplicity, on the plane) a given density function ρ and the total "mass" in a region Ω:

$$M(\Omega) = \int_\Omega \rho(x,y)dxdy.$$

We want to see how M depends on Ω. (Assume that the regions are "convex-like" and can be deformed smoothly without introducing intersections.)

(a) First take the special case that Ω is the area between the x-axis and the graph of a function $\eta = \eta(x)$:

$$\Omega = \{\ (x,y)\ \mid\ a \le x \le b,\ 0 \le y \le \eta(x)\ \},$$

and consider the corresponding functional

$$\mathcal{L}(\eta) = M(\Omega).$$

Determine the first variation and show that the variational derivative of $\mathcal{L}$ for variations of the domain described by a variation of the function η is given by

$$\delta\mathcal{L}(\eta) = \rho\ |_{y=\eta(x)} \equiv \rho(x,\eta(x)).$$

(b) Now, more generally, describe a variation of the boundary $\partial\Omega$ by a "normal" displacement σ (defined on the boundary). Determine the first variation of M.
Can you find an expression for the variational derivative of M?
Verify the formula for the case of a radial deformation of a circular domain (and $\rho = 1$).

(c) Show that the more general result specializes to the case of changing the graph that determines the boundary. (Relate a variation η and the normal displacement σ in this case.)

Bibliography

[1] D. Adytia, Simulations of short-crested harbour waves with variational Boussinesq modelling. In: *Proceedings ISOPE*, 2014, 912-918.

[2] D. Adytia & Lawrence, Fully nonlinear dispersive HAWASSI-VBM for coastal zone simulations, OMAE 2016

[3] D. Adytia, M. Woran & E. van Groesen, Effect of a possible Anak Krakatau explosion in the Jakarta Bay, *Proceedings Basic Science International Conference 2012*, Malang Indonesia, K1-5, ISBN 978-979-25-6033-6

[4] D. Adytia, M. Ramdhani & E. van Groesen, Phase resolved and averaged Wave Simulations in Jakarta Harbour, *Proceedings 6th Asia-Pacific Workshop on Marine Hydrodynamics*-APHydro2012, Johor Baru Malaysia, 3-4 September 2012. pp. 218-223.

[5] D. Adytia, *Coastal zone simulations with Variational Boussinesq Modelling*, PhD-Thesis University of Twente, Netherlands, May 2012

[6] D. Adytia and E. van Groesen, Optimized Variational 1D Boussinesq modelling of coastal waves propagating over a slope, *Journal Coastal Engineering*, 64 (2012) pp. 139-150

[7] D. Adytia and E. van Groesen, The variational 2D Boussinesq model for wave propagation over a shoal, *International Conference on Developments in Marine CFD*, 18 - 19 November 2011, Chennai, India; RINA, ISBN No : 978-1-905040-92-6, p.25-29.

[8] D. Adytia & E. van Groesen, Variational Boussinesq model for simulations of coastal waves and tsunamis, *Proceedings of the 5th International Conference on Asian Pacific Coasts*, (APAC2009) 13-16 October 2009 Singapore 9ed: Soon Keat Tan, Zhenhua Huang]; World Scientific 2010, ISBN-13 978-981-4287-94-4, Volume 1 (ISBN-13 978-981-4287-96-8), pages: 122-128.

[9] D. Adytia, A. Sopaheluwakan & E. van Groesen, Tsunami waveguiding: phenomenon and simulation above synthetic bathymetry and Indonesian coastal area, *Proceedings of International Conference on Tsunami Warning* Bali, Indonesia, November 12-14, 2008

[10] L.J.F. Broer, On the Hamiltonian theory of surface waves, *Appl. Sci. Res.* **30** (1974) 340.

[11] L.J.F. Broer, E. van Groesen & J.M.W. Timmers, Stable model equations for long water waves, *Appl. Sci. Res.* 32 (1976) 619-636.

[12] R. Courant & D. Hilbert, *Methods of Mathematical Physics*, Interscience, 1953

[13] P.R. Garabedian, *Partial differential equations*, John Wiley, 1963

[14] I.M. Gelfand & S.V. Fomin, *Calculus of Variations*, Prentice Hall, 1963

[15] Y. Goda, *Random seas and design of maritime structures*, World Scientific, 3th edition, 2010

[16] H.H. Goldstein, *A history of the Calculus of Variations*, Springer 1980

[17] E. van Groesen & E.M. de Jager, *Mathematical structures in continuous dynamical systems*, North-Holland, 1994

[18] E. van Groesen & G. Klopman, Dispersive effects in tsunami generation; *Proceedings Rogue Waves* 2004, Brest France.

[19] E. van Groesen, Variational Boussinesq Model, part 1: Basic equations in Cartesian coordinates; Technical Report LMI-GeoMath-06/02, 27 January 2006; revised 9 March 2006.

[20] E. van Groesen & J. Molenaar, *Continuum modeling in the Physical Sciences*, SIAM , Mathematical Modeling and Computation 13, 2007, p. 236; ISBN-13: 978-0-898716-25-2

[21] E. van Groesen & Andonowati, Variational derivation of KdV-type of models for surface water waves, *Physics Letters A* **366**(2007)195-201.

[22] E. van Groesen, D. Adytia & Andonowati, Near-coast tsunami waveguiding: phenomenon and simulations, *Natural Hazards and Earth System Sciences*, **8**(2008) 175-185.

[23] E. van Groesen, Andonowati, L. She Liam & I. Lakhturov, Accurate modelling of uni-directional surface waves, *Journal of Computational and Applied Mathematics*, **234** (2010) 1747-1756

[24] E. van Groesen & Andonowati, Time-accurate AB-simulations of irregular coastal waves above bathymetry, *Proceedings of the Sixth International Conference on Asian and Pacific Coasts* (APAC 2011) December 14 – 16, 2011, Hong Kong, China, World Scientific ISBN: 978-981-4366-47-2, pp.1854-1864

[25] E. van Groesen, T. Bunnik & Andonowati, Surface wave modelling and simulation for wave tanks and coastal areas, International Conference on Developments in Marine CFD, 18 - 19 November 2011, Chennai, India, RINA, ISBN: 978-1-905040-92-6, p. 59-63

[26] E. van Groesen & Andonowati, Fully dispersive dynamic models for surface water waves above varying bottom, Part 1: Model equations, *Wave Motion* 48 (2011) 657-666

[27] E. van Groesen & I. van der Kroon, Fully dispersive dynamic models for surface water waves above varying bottom, Part 2: Hybrid spatial-spectral implementations, *Wave Motion* 49 (2012) 198-211

[28] E. van Groesen & Andonowati, Hamiltonian Boussinesq formulation of Wave-Ship-Interactions, submitted 2015

[29] L.H. Holthuijsen, *Waves in oceanic and coastal waters*, Cambridge University Press, 2007

[30] G. Klopman, *Variational Boussinesq modelling of surface gravity waves over bathymetry*, PhD thesis University of Twente, Netherlands, May 2010

[31] G. Klopman, E. van Groesen & M.Dingemans, A variational approach to Boussinesq modelling of fully non-linear water waves, *Journal Fluid Mechanics* **657** (2010) 36-63

[32] G. Klopman, M.W. Dingemans & E. van Groesen, A variational model for fully non-linear water waves of Boussinesq type; in *Proceedings of 20th International Workshop on Water Waves and Floating Bodies*, Spitsbergen, Norway, 29 May - 1 June 2005, http://www.iwwwfb.org/Abstracts/iwwwfb20/iwwwfb20_30.pdf.

[33] G. Klopman, M. Dingemans & E. van Groesen, Propagation of wave groups over bathymetry using a variational Boussinesq model, *Proceedings Int. Workshop on Water Waves and Floating Bodies*, (eds: Šime Malenica and Ivo Senjanović), Plitvice, Croatia, April 2007, pp125-128; http://www.iwwwfb.org/Abstracts/iwwwfb22/iwwwfb22_32.pdf.

[34] W. Kristina. O. Bokhove & E. van Groesen, Effective coastal boundary conditions for tsunami wave run-up over sloping bathymetry, *Nonlin.Processes GeoPhys*, 21(2014) 987-1005

[35] R. Kurnia, *HAWASSI-AB modelling and simulation of fully dispersive nonlinear waves above bathymetry*, PhD thesis University of Twente, Netherlands, February 2016

[36] R. Kurnia, Simulations of extreme wave run-up on a vertical wall by Analytic Boussinesq model, *OMAE* 2016

[37] R. Kurnia & E. van Groesen, Localization in Spatial-Spectral Method for Water Wave Applications, in R.M. Kirby et al. (eds.), Spectral and High Order Methods for Partial Differential Equations ICOSAHOM 2014, Lecture Notes in Computational Science and Engineering 106, 305-313, Springer 2015

[38] R. Kurnia & E. van Groesen, Localization for spatial-spectral implementations of 1D Analytic Boussinesq equations, submitted

[39] R. Kurnia & E. van Groesen, High Order Hamiltonian Water Wave Models with Wave-Breaking Mechanism, *Coastal Engineering* 93(2014) 55–70

[40] R. Kurnia, T. van den Munckhof, C.P. Poot, P. Naaijen, R.H.M. Huijsmans & E. van Groesen, Simulations for design and reconstruction of breaking waves in a wavetank. *OMAE 2015*, vol 2-41633 (7pages)

[41] R. Kurnia & E. van Groesen, Design of Wave Breaking Experiments and A-Posteriori Simulations, Memorandum 2042, (January 2015), http://www.math.utwente.nl/publications ISSN 1874-4850

[42] R. Kurnia & E. van Groesen, Spatial-spectral Hamiltonian Boussinesq wave simulations, *Advances in Computational and Experimental marine hydrodynamics*, VOL. 2 -41633 Conference Proceedings, 2014, pp. 19-24, ISBN: 978-93-80689-22-7

[43] Ivan Lakhturov, *Optimization of Variational Boussinesq Models*, PhD thesis University of Twente, Netherlands, November 2012

[44] I. Lakhturov, D. Adytia & E. van Groesen, Optimized Variational 1D Boussineq modelling for broad-band waves over flat bottom, *Wave Motion*, 49 (2012)309-322

[45] I. Lakhturov & E. van Groesen, Optimized Variational Boussinesq Modelling; part 1: Broad-band waves over flat bottom, Feb 2010, Memorandum http://eprints.eemcs.utwente.nl/17397/

[46] H. Lamb, Hydrodynamics, Dover 1932 (sixth edition)

[47] L.D. Landau & E.M. Lefshetz, Fluid Mechanics, vol.6, Course of Theoretical Physics, Pergamon Press, 1989

[48] A.L. Latifah & E. van Groesen, Localized Coherence of FreakWaves, submitted 2016
[49] A.L. Latifah & E. van Groesen, Coherence and Predictability of Extreme Events in Irregular Waves, *Nonlin. Processes Geophys*, 19 (2012)199-213, ISSN 1023-5809
[50] Lie S Liam, *Mathematical modelling of generation and forward propagation of dispersive waves*, PhD thesis University of Twente, Netherlands, May 2013
[51] Lie S Liam, D. Adytia & E. van Groesen, Embedded wave generation for dispersive surface wave models, *Ocean Engineering* 80(2014) 73-83
[52] L. She Liam & E. van Groesen, Variational derivation of KP-type equations, *Physics Letters A*, **374**(2010) 411-415
[53] J.C. Luke, A variational principle for a fluid with a free surface, *J. Fluid Mech.* **27** (1967) 395-397.
[54] C.C. Mei, The applied dynamics of Ocean surface waves, World Scientific, 1994
[55] J.W. Miles, On Hamilton's principle for surface waves, *J. Fluid Mech.* **83** (1977), 153-158
[56] J.J. Stoker, *Water waves*, Interscience Publishers, 1957
[57] G.B. Whitham, *Linear and nonlinear waves*, Wiley-Interscience, 1974
[58] A.P. Wijaya & E. van Groesen, Significant wave height retrieval from synthetic radar images, Ocean Engineering 114(2016)204-215
[59] A.P. Wijaya & E. van Groesen, Significant wave height retrieval from synthetic radar images, *Proceedings of the 11th International Conference on Hydrodynamics* (ICHD 2014) October 19 – 24, 2014, Singapore
[60] A.P. Wijaya, P. Naaijen, Andonowati & E. van Groesen, Spatial evolution scenario from synthetic radar images to sea states near the radar, 2015 in press *Ocean Engineering*
[61] A.P. Wijaya, Towards nonlinear wave reconstruction and prediction from synthetic radar images, *OMAE* 2016
[62] V.E. Zakharov, Stability of periodic waves of finite amplitude on the surface of a deep fluid, *J. Appl. Mech. Tech. Phys.* **2** (1968) 190-194.
[63] E. Zauderer, *Partial differential equations*, 2nd edition, John Wiley, 1989